The Rayleigh–Taylor Instability Among the Stars

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The instability of the interface separating a denser fluid from a lighter one below it has applications to the surroundings of massive stars – both when they are born and when they die.

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In school physics, a pencil standing vertically on its point is a standard example of unstable equilibrium – the slightest push in any direction, and it would fall to the ground. One would think that the study of such phenomena would be only of academic interest. The article by Chirag Kalelkar, in the classroom section of this issue, demonstrates the Rayleigh–Taylor instability in the laboratory and shows how interesting and rich this phenomenon can be. This is a good opportunity to introduce the same theme in a different context, and on a grander scale – the energetic phenomena around massive stars. On this scale, surface tension and viscosity do not play a role. One should also bear in mind the different style of working often adopted in astrophysics. Rough estimates are first made based on simplified models to see if a given phenomenon is likely to occur in a certain situation. Once this is established, elaborate numerical simulations can be carried out and compared with the observations. This article will only deal with simplified physical pictures and should be read keeping in mind that the rigorous treatment is needed for real-world applications.

Rayleigh is the famous 19th century classical physicist known for his explanation of the blue of the sky and his monumental work on sound. Incidentally, 'Rayleigh' is a title he assumed after becoming a Lord in the English tradition – his real name was J W Strutt which lacks the majesty of his assumed title. In any case, in 1883, he asked himself the following question – what happens if a heavy fluid sits on top of a light one? (*Figure* 1(a)). We will start to answer this question by considering the opposite situation – a light fluid on top of a heavy one (*Figure* 1(b)), say oil on water.

Keywords
Rayleigh–Taylor instability, Kelvin–Helmholtz instability, effective gravity.

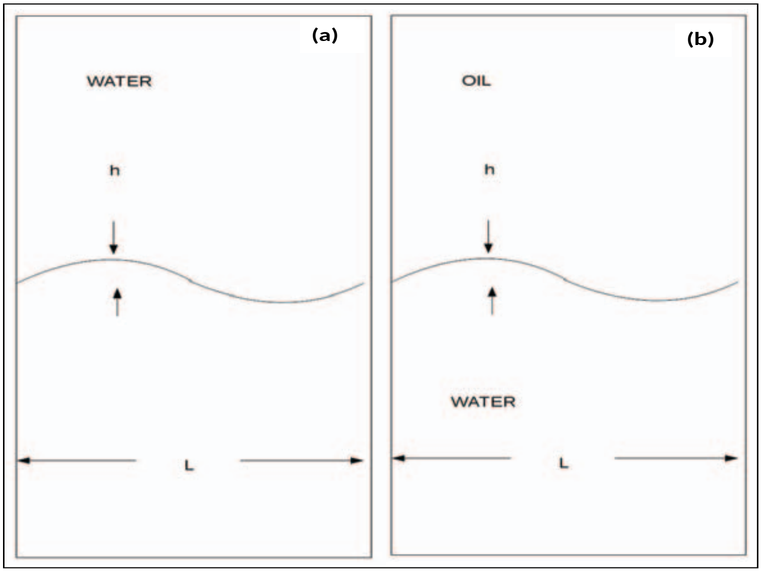

Figure 1. **(a)** Disturbing the interface between a heavy fluid on top and a lighter fluid below. This is an unstable situation, and the height of the disturbance grows exponentially at early times. **(b)** A sinusoidal disturbance of height h on an interface separating a lighter fluid (oil) on top of a heavier fluid (water) is stable, and hence undergoes oscillatory motion after it is disturbed.

We should emphasize that both situations are in equilibrium when the interface is horizontal. The pressure at each level equals the total weight of the column of fluid over a unit area. Hence, each fluid element has zero net force on it when there is no disturbance.

Figure 1(b) includes a small sinusoidal displacement $\delta z(x) = h \sin kx$ of the boundary between the two fluids in a deep tank. The length of the tank 'L' has been chosen so that one wave fits into it, so $kL = 2\pi$. The width of the tank, perpendicular to the plane of the figure, is denoted by 'w'. We estimate the effect of the deformation on the potential energy. *Figure* 1(b) shows how a volume of the lower fluid of the order $h(L/2)w$, has been raised to a height of the order of h. One could make more accurate estimates, but this is all we need. The potential energy of the lower fluid has increased by $\delta V_l \sim \rho_l Lwh^2$. At the same time, the potential energy of the upper fluid has changed in the opposite direction by $\delta V_u \sim -\rho_u Lwh^2$. In this case, a volume of fluid of the order wLh of density ρ_u has been moved downwards by an amount of the order of h. The total change in potential energy due to the displacement of the interface is therefore, $\delta V \sim (\rho_l - \rho_u) wLh^2$. The kinetic energy associated with this movement is of the order,

$$(\rho_u + \rho_l)\, wL^2 \left(\frac{\mathrm{d}h}{\mathrm{d}t}\right)^2 .$$

There is an important point to be noted here – why have we moved a larger volume wL^2 to calculate the kinetic energy, compared to the smaller volume wLh which we used in calculating the potential energy? The reason is that the fluid is assumed to be incompressible. The up-down motion of the interface necessarily has to disturb the fluid, all the way down to a depth of the order of L, if it is to transfer the fluid over a distance of L. We are assuming here that the depth of the tank is greater than L. This holds for the fluid on both sides of the boundary.

We can now see an analogy to the harmonic oscillator, for which the potential and kinetic energies are proportional to kx^2 and $m\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2$. We therefore conclude that such a displacement $h(t)$ will oscillate at a frequency given by the analogue of $\omega^2 = k/m$ for the harmonic oscillator. This would now be given by,

$$\omega^2 \sim \frac{(\rho_l - \rho_u)}{(\rho_l + \rho_l)}\, g/L .$$

We have thus estimated the frequency of oscillation of the stable interface between a lighter fluid sitting above a heavier fluid. The result of a proper calculation is,

$$\omega^2 = \frac{(\rho_l - \rho_u)}{(\rho_l + \rho_l)}\, g(2\pi/L) .$$

This formula describes the propagation of deep water waves in the case when the upper medium is air. We have assumed that the depth is greater than the wavelength L here, and we have analyzed a standing wave.

This is a positive quantity because $\rho_l > \rho_u$. This formula describes the propagation of deep water waves in the case when the upper medium is air. We have assumed that the depth is greater than the wavelength L here, and we have analyzed a standing wave. But the relation between frequency and wavelength works for propagating waves as well. This problem was tackled by Laplace, and brought to final form by Airy in 1840, well before Rayleigh's work.

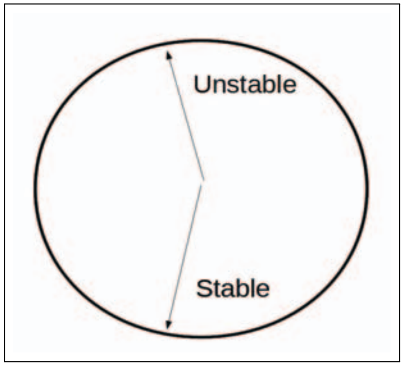

Figure 2. A mass confined to a circle by a rigid spoke undergoes stable oscillations near the bottom. But the displacement grows near the top which is an unstable position.

Coming back to our original problem, what happens when the heavier fluid is above the lighter fluid, as in *Figure* 1(a)? The calculation of the kinetic energy remains the same, but the potential energy now changes sign. We can approach the problem by drawing an analogy with the simple pendulum The one modification is that we use a rigid spoke, rather than a string, connecting the bob and the suspension (*Figure* 2). But now, the potential energy change due to a small displacement has to be negative and proportional to the square of the displacement for small displacements. This is exactly the situation near the top of the circle along which the bob moves in *Figure* 2. Clearly, the bob will fall down on one side or the other – any small displacement will continue to grow[1].

[1]In laboratory situations, at short wavelengths, a third contribution to the energy comes from the surface tension between the two fluids. This is always positive since the area of the interface increases when it is disturbed. This can stabilize an otherwise unstable situation (See Chirag Kalelkar's article).

There is also a more mathematical way of viewing this situation. In the stable case, oscillations are described by trigonometric functions which can be written as exponentials of imaginary quantities. For example, $\cos(\omega t) = (\exp(i\omega t) + \exp(-i\omega t))/2$. Here ω is a real quantity since we found a positive value for ω^2. Now in the unstable case, we have a negative value for ω^2, which means that ω is a pure imaginary quantity, which we denote by $\pm i\alpha$. That means that our earlier solution will now contain $\exp(\pm\alpha t)$. Unless the initial conditions are very special, the growing exponential – positive sign in the exponent – will not be zero and will dominate at large times. The quantity α is, therefore, to be interpreted as a growth rate. In a time $1/\alpha$, the amplitude grows by a factor, $e = 2.71828....$

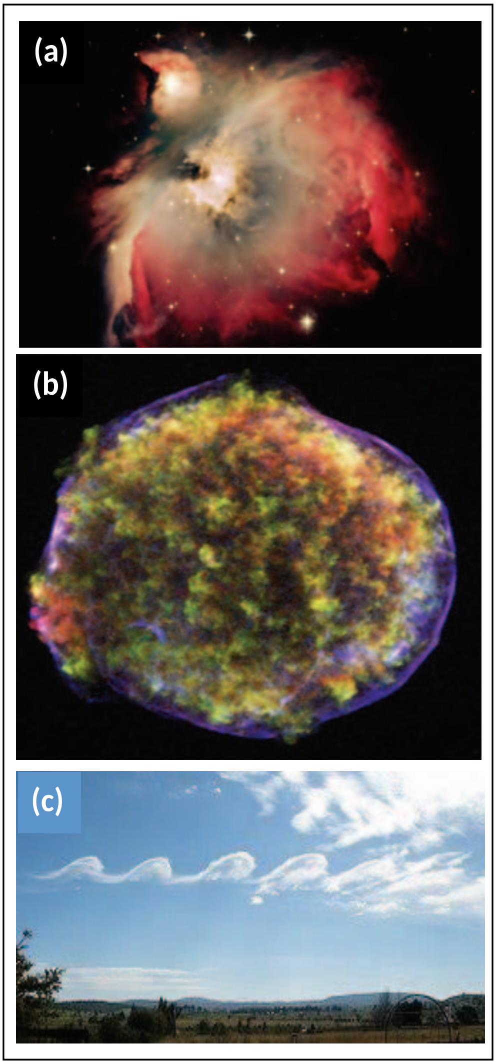

Figure 3. **(a)** Newly formed stars in the Orion region. The dark features could be a product of the Rayleigh–Taylor instability (Image courtesy: NASA Hubble space telescope). **(b)** The remnant of the supernova which was seen by the astronomer Tycho Brahe in 1572. Some of the features seen in the gas are attributed to the Rayleigh–Taylor instability (Image courtesy: NASA, Chandra observatory). **(c)** Our own atmosphere has layers in relative motion. This picture shows that the interface has rolled up, as made visible by clouds. This is one of the early phases of the Kelvin–Helmholtz instability. (Source: http://en.wikipedia.org/wiki/Image: Wavecloudsduval.jpg)

In both the unstable and stable cases, the analysis so far, based on the harmonic oscillator analogy, assumes that the amplitudes are small. In the stable case, this assumption is at least internally consistent – if we start small, it stays small. However, in the unstable case, it will sooner or later become large, and the so-called 'linear analysis' will fail. So, all serious applications of instability analysis take into account what happens when the amplitude grows large. Quite often, this is best done by numerical simulations. One exception is a discussion of the instability at large

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amplitudes by Fermi (see suggested reading). This is carried out in his characteristic style of insightful simplification, backed up by analysis and numerical estimates.

[2]See *Resonance*, Vol.9, No.10, 2004.

When does the famous fluid scientist G I Taylor[2] come into the picture? Surprisingly, in the early 1950s. He realized that the situation envisaged by Rayleigh would arise whenever a light fluid pushes against a heavier fluid and accelerates it. In this accelerated frame of reference, we can work with an effective gravity which is minus the acceleration. (The textbook example is that if we are placed in an elevator accelerated upwards, we feel as if we are being pushed downwards. An object released drops to the floor, exactly as if gravity were present.) This effective gravity points towards the lighter fluid, which is, therefore, 'below' the heavier one – an unstable situation. Equally, if a heavy fluid is decelerated by colliding with a light fluid, the effective gravity points towards the lighter fluid, and we have an instability.

Taylor's own interest was possibly in the physics of atomic explosions in the atmosphere, which he was working on at that time. His work has been utilized by astrophysicists to understand some remarkable features seen in the gas surrounding young stars. An example is the Orion Nebula (*Figure* 3(a)) with recently formed stars, much more massive than the Sun, shining into the surrounding gas. These stars are formed from dense molecular gas. Once they start nuclear reactions at the center, they emit strong ultraviolet radiation. Some of them also send out matter in the form of a more or less spherically symmetric wind (even our Sun has wind, but it is relatively mild). The radiation and the wind push against the surrounding gas, accelerating it outwards. We now have the situation of a light fluid accelerating a heavy fluid. What we are witnessing in the Orion Nebula is a very late stage of the instability. The denser outer fluid here is not a gentle sinusoidal wave – it has sharpened into 'elephant trunks'. This explanation has been explored using numerical simulations and is believed to be broadly correct.

Similar ideas have been applied to supernova explosions. The shell of material thrown out during the explosion of a star is denser than the surrounding interstellar material, which decelerates it. Now, the effective gravity is outwards but again points from the denser to the less dense medium, so that the same instability operates. The detailed astrophysics is beyond the scope of this piece, but the picture is worth viewing (*Figure* 3(b)).

In the late stage of the Rayleigh–Taylor instability, the light, and the heavy fluids are rushing past each other. This brings in the Kelvin–Helmholtz instability[3]. Again, Kelvin was just plain William Thomson till he became a Lord! This instability can thoroughly mix the two fluids as the interface widens in the late stages. This mixing can play an important role in the astrophysics of the gas around young stars[4].

[3]The Inveterate Tinkerer: 8. Kelvin–Helmholtz Instability, Chirag Kalelkar, *Resonance*, Vol.22, No.10, pp.955–960, 2017.

[4]Note that the KH instability taps excess kinetic energy and hence does not require a heavier fluid above a lighter one.

In the Rayleigh–Taylor instability, we see a fluid almost at rest initially, set into violent motion in the later stages. Clearly, this energy comes from the potential energy of the heavy fluid descending into the light fluid. The Kelvin–Helmholtz instability pertains to two fluids moving at different speeds, separated by an interface (which need not be thin in general). In this case, the system has kinetic energy of the ordered relative motion. Some of this is converted to the kinetic energy of motion perpendicular to the interface, as the two layers penetrate each other. Initially, this motion could be orderly – see the beautiful *Figure* 3(c). But ultimately, this does become random. This kind of general picture, tapping a source of excess energy and using it to generate motions, which ultimately become random, is useful in rationalizing many kinds of instability, once found. However, it is not a substitute for the hard work of investigating each problem numerically and experimentally!

Suggested Reading

[1] **Rajaram Nityananda, Fermi and the Art of Estimation, *Resonance*, Vol.19, No.1, pp.73–81, 2014.**

[2] **Arnab Rai Choudhuri, *The Physics of Fluids and Plasmas: An Introduction for Astrophysicists*, Chapter 7, Cambridge University Press, 1998.**

[3] **Enrico Fermi, *Collected Papers – Vol.II, US 1939–1954*, 244. Taylor Instability of an Incompressible Liquid, Part I of Document AECU-2979 (September 4, 1951), pp.816–820, The University of Chicago Press, 1965.**

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