The Bondi Approach to the Lorentz Transformation

From Alpha to Delta

Rajaram Nityananda

This article summarises a very elegant way of teaching the Lorentz transformation, which was introduced by Hermann Bondi (see *Resonance*, March 2016 for more about him). The starting point of this approach is that the best way of assigning space and time coordinates to a distant event is by radar, i.e., by transmitting and receiving light signals, and noting the times of departure and arrival. There is no need for the rigid rod. Bondi's method, which he called the *k*-calculus, makes good use of the Doppler effect (hence the delta in the title). For students who have learnt special relativity in the standard way, this approach provides a different view of the subject.

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1. Determining Co-ordinates by Radar – Spare the Rod!

All textbook and classroom expositions of special relativity start with the familiar set-up of two observers, A and B moving relative to each other along the x-axis. In the Western tradition, these are Alice and Bob. We in India, particularly Punjab, can do better. Amandeep and Balwinder can be interpreted in four ways as far as gender is concerned, by adding Singh or Kaur as one chooses! The co-ordinate system/frame of reference of A is traditionally called S (co-ordinates of an event E as measured by A are x and t). B's frame of reference is called S$'$ (co-ordinates of the same event E as measured by B are x' and t').

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Purely for convenience, A and B choose their origin of space and time, O, to be the event of their passing each

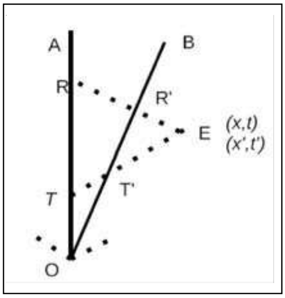

Figure 1. A space-time diagram with the *x*-axis horizontal and the *t*-axis vertical. The world line of A is therefore a vertical solid line and that of B slopes to the right as this observer moves away from A at a speed *v*. Light signals exchanged between these observers and an event E are shown as thick dotted lines, with T standing for transmission and R for reception.

other at a relative velocity v along the x-axis. The event E, which they are both studying, is at a general place on the x-axis, at a general time, as shown by the standard space–time diagram of *Figure* 1. The thick lines are the trajectories, also called 'world lines' in special relativity, of the two observers. They are drawn as if A is at rest, but of course, this is just one way of drawing them, A is not privileged with respect to any other observer. The thick dotted lines TT′ER′R are light (or radio) signals, which travel to the event E, and return to the observer. The time of transmission by A is T and the time of reception back at A is R. The idea is that A will be able to assign the co-ordinates x and t of the event E as follows.

It is clear from *Figure* 1 that the travel time from the observer A to E is x/c. Therefore,

$$T = t - \frac{x}{c} \text{ and } R = t + \frac{x}{c}.$$

This can also be written as

$$x = \frac{c(R-T)}{2}, \quad t = \frac{(R+T)}{2}.$$

This second form is intuitively reasonable as well and is the basis for how radar works. The time difference $R-T$ for the two-way journey is $2x/c$, while the time of arrival of the signal at the event E is the average of the transmission and reception times. Einstein's postulate that the speed of light is c for observer A (and in fact for any other observer!) has been used. Of course, it is meaningless to ask if A is moving or not.

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Now, we note that observer B can use the same light signals, both outgoing and return. One could imagine that B simply notes the time at which the outgoing signal passes, and calls it T'. Likewise, B notes when the return signal passes, and calls it R'. Then the co-ordinates of the event E, as assigned by radar, according to B, are given by the same equations as for A, but now with primes on them. This is where we use the first postulate of special relativity – complete democracy between A and B.

$$T' = t' - \frac{x'}{c}, \quad R' = t' + \frac{x'}{c}$$

and

$$x' = \frac{c(R' - T')}{2}, \quad t' = \frac{(R' + T')}{2}.$$

2. Enter Doppler

The crucial step comes now. The Doppler effect enables us to relate T to T' and R to R'. For this purpose, we have to imagine two more light signals passing from A to B, and B to A, at the time that they were coincident. These of course travel zero distance in zero time, but are shown as short dotted lines emanating from O in the figure. We can now say that two light signals were emitted by A at times 0 and T, and received by B at times 0 and T'. The different separation in time of these two signals as perceived by the two observers is nothing but the Doppler effect. We usually think of the separation in arrival time of two successive wave crests at B, divided by the separation at A, as the Doppler factor, which we denote by δ. The same factor will give us the ratio of T' to T. Since B is moving away from A, we expect this to be greater than unity. The time period increases, or the frequency decreases, in this case of two observers moving away from each other.

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The same Doppler factor applies when we try to relate R to R'. Note that R' is the time separation of the two light beams when they pass B, and R when they pass A. So, we now get

$$\delta = \frac{T'}{T} = \frac{R}{R'}.$$

There is a further assumption of symmetry here, that the Doppler factor going from S to S' is the same as that going from S' to S. Otherwise, physics would be different for light traveling in two opposite directions! Using the earlier expressions for R, T, R', T' in terms of the co-ordinates of the event, we get the Lorentz transformation in a rather nice form.

$$t' - \frac{x'}{c} = \left(t - \frac{x}{c}\right)\delta,\ t' + \frac{x'}{c} = \left(t + \frac{x}{c}\right)\delta^{-1}.$$

3. A Matter of Rapidity

Solving for x' and t' in terms of x and t, we get something looking more like the Lorentz transformation, especially if we put

$$\delta = \exp(\alpha)\,.$$

We then get, after a few steps which are omitted,

$$x' = x\cosh(\alpha) - ct\sinh(\alpha)\,,$$

$$ct' = ct\cosh(\alpha) - x\sinh(\alpha).$$

But where is the relative velocity v of the two frames? Simple, just put x' equal to zero, so that one is sitting on the origin of S. The ratio of x to ct gives

Since we are expressing everything in terms of α, we should have a name for it. The standard term is 'rapidity'. Unlike velocity which is limited by c, rapidity goes from $-\infty$ to $+\infty$. It has the nice property that it adds under Lorentz transformations in the same direction.

$$v/c = \tanh(\alpha)\ \equiv\ \beta\,.$$

We have followed tradition and denoted the ratio of the speed v (velocity of B with respect to A) to that of light, by β. The Lorentz factor γ giving time dilation and length contraction is given by

$$\frac{1}{\sqrt{(1 - v^2/c^2)}} = \frac{1}{\sqrt{(1 - \tanh^2\alpha)}} = \frac{1}{\operatorname{sech}\alpha} = \cosh\alpha.$$

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it adds under Lorentz transformations in the same direction. This could be checked in the usual way by introducing a third frame S″, moving with respect to S′, combining the two Lorentz transformations, and using identities between hyperbolic functions.

The factor f can only depend on the relative velocity and not even on its sign, otherwise we would be favouring right over left, or vice versa!

Because of α being connected to the Doppler effect, we can however see the result directly. From Bondi's treatment, it is obvious that the Doppler factor is multiplicative, i.e., the δ going from S to S″ is the product of the δ's going from S to S″ and from S′ to S″. Clearly, the logarithm of δ, which we called α, is going to be additive on these physical grounds.

The fact that rapidity is additive is equivalent to the famous Einstein velocity addition formula. If we write $\alpha_{\mathrm{AC}} = \alpha_{\mathrm{AB}} + \alpha_{\mathrm{BC}}$, then taking the hyperbolic tangent on both sides, we have

$$\beta_{\mathrm{AC}} = \tanh\alpha_{\mathrm{AC}} = \tanh(\alpha_{\mathrm{AB}} + \alpha_{\mathrm{BC}})$$

$$= \frac{(\tanh\alpha_{\mathrm{AB}} + \tanh\alpha_{\mathrm{BC}})}{(1 + \tanh\alpha_{\mathrm{AB}}\tanh\alpha_{\mathrm{BC}})} = \frac{\beta_{\mathrm{AB}} + \beta_{\mathrm{BC}}}{1 + \beta_{\mathrm{AB}}\beta_{\mathrm{BC}}}. \qquad (1)$$

The relationship between the transverse co-ordinates, $y' = y, z' = z$ follows in the same way as in the usual textbook treatment, so is just sketched here. Let the transverse co-ordinates be related by a factor f, i.e., $y' = fy, z' = fz$. The factor f can only depend on the relative velocity and not even on its sign, otherwise we would be favouring right over left, or vice versa!. So we get f^2 when we go from S to S′ and come back. So f ends up being unity.

4. An Additive Formula for Aberration

With the three-dimensional Lorentz transformation in hand, one can now consider a light beam travelling at an angle θ to the x-axis, and calculate the Doppler effect (change in frequency) and aberration (change in direction when going from the S frame to the S′ frame). The standard formula reads

$$\cos\theta' = \frac{(\cos\theta - \beta)}{(1 - \beta\cos\theta)}\,,$$

where θ and θ' are the angles made by the same light ray as viewed in the two frames S and S$'$. As an interesting exercise, students can try and rewrite the standard aberration formula in an additive form! To do this, we need to introduce $\ln(\cot(\theta/2))$ and denote it by (what else?) ϵ. The aberration formula now reads (proof an exercise)

$$\epsilon' = \epsilon - \alpha\,.$$

Who would think of subtracting the rapidity of B with respect to A from the logarithm of the cotangent of half the angle that a ray makes with the x-axis in one frame? Actually, this is nothing but the additivity of rapidity in disguise. For a beam travelling along the x-axis, in either direction, it continues to be along the x-axis; $\theta = 0, \pi$ (so $\epsilon = \infty, -\infty$) in any frame of reference. In the general case, the x-component of the velocity of light is $c\cos\theta$ in the S frame, and hence the corresponding rapidity is $\epsilon = \tanh^{-1}(\cos\theta)$. Using the formula $\tanh^{-1}(x) = (1/2)\ln((1+x)/(1-x))$ and simplifying, we get $\epsilon = \ln(\cot(\theta/2))$, the quantity we had introduced. We therefore get the new (primed) rapidity, ϵ' by subtracting the rapidity α of S$'$ from ϵ.

Bondi's approach shows how much he was ahead of his time, in building up special relativity using just clocks and banishing the rod! I hope that this concise summary will encourage students and teachers to go to his 1964 book which has many original ways of understanding relativistic phenomena, including those we have not discussed like the variation of mass with speed.

5. Conclusion

In today's world, precise positions and times on earth are given by receiving radio signals broadcast at precise times from satellites – the GPS system installed in almost every mobile telephone. The distance to spacecraft and to solar system bodies is routinely determined by radar – our knowledge of the solar system geometry is entirely based on time measurements on earth. Bondi's approach shows how much he was ahead of his time, in building up special relativity using just clocks and banishing the rod! I hope that this concise summary will encourage students and teachers to go to his 1964 book which has many original ways of understanding relativistic phenomena, including those we have not discussed like the variation of mass with speed. I apologise for replacing his Doppler factor k with δ for two reasons – this notation is common in astrophysics, and the temptation to use the first five letters of the Greek alphabet was irresistible!

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Suggested Reading

[1] Hermann Bondi, *Relativity and Common Sense: A New Approach to Einstein*, Dover (1980), reprint of the original 1964 edition.