Review

Review of G H Hardy's A MATHEMATICIAN'S APOLOGY

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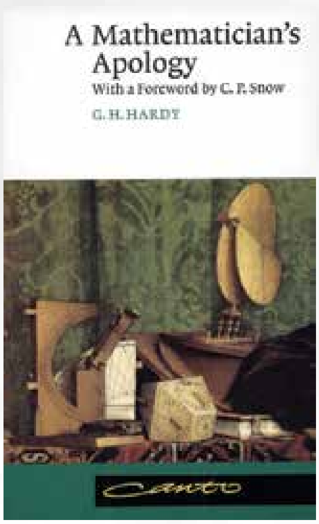

Why an apology?

G. H. Hardy (1877–1947), a mathematician known for his deep contributions to Analysis and Number Theory, wrote this book in 1940, when he was 62 years old. It is more a long essay than a book, and remains interesting to this day, more than 75 years after it was published.[1] In what way is it relevant to us, especially mathematics teachers and students of today? To Indians, Hardy is best known for his discovery of Srinivasa Ramanujan and subsequent collaboration with him. For many all over the world, Kanigel's book *The Man Who Knew Infinity*, and a popular Hollywood film released last year based on the book, have made Hardy famous among the general public. The gaunt solitary Cambridge don that Hardy was, entirely intellectual, cricket loving, and awkward in social interactions, fits the popular image of a professor and mathematician. The style of the book reinforces the image in many ways.

The essay raises the question:

I shall ask, then, why is it really worth while to make a serious study of mathematics? What is the proper justification of a mathematician's life?

[1] In 1967, Hardy's book was re-published with a long foreword by C. P. Snow; it's a delightful biography of Hardy, and discusses in detail Hardy's collaboration with Littlewood and Ramanujan.

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In trying to answer these questions, he raises more: what criteria go into deciding what constitutes *good* mathematics? Should we distinguish pure and applied mathematics? Is it important that mathematics be *useful*? The essay is a long argument that attempts to answer these questions honestly and sincerely, in an autobiographical manner. It is an apology, not in the sense of seeking society's forgiveness, but in providing a detailed justification to society, for not only seeking to spend one's productive life on what seems a useless self-indulgent preoccupation, but also asking that society provide support for it. In the 21st century, when it is widely acknowledged that mathematics has tremendous life changing applications, one may be tempted to think that mathematics needs no justification. But one has to only look at modern science policy and its obsession with 'thrust areas' and 'applicability' to realise that "pure" mathematics cannot take public financing for granted. The recent stress on 'STEM education'[2] is indeed positive on mathematics education but it envisions mathematics as enhancing 'employability' and contributing to industrial development. It is hardly clear whether pure mathematics with no apparent applications would be considered justifiable in such a vision. Therefore, some justification is worthwhile.

Content

Quickly, Hardy disposes of two reasons why a person might do something all his life. The first is, "I do X because X is the only thing I do well." The second is, "This just came my way, so I do it." These might suffice for individuals, but perhaps not for the social worth of an activity. In essence, Hardy wishes to consider only those who do something well, and then ask, is the activity worth doing well? In that sense, why is mathematics worthwhile? I do not wish to go through all the hypotheses proposed by Hardy, with arguments and counter-arguments. In brief, and not necessarily in the order in which Hardy discusses them, he asserts the following.

1. Mathematical achievement is *permanent*. Unlike all other forms of knowledge, there is a certainty and permanence to mathematical truth, which makes pursuit of mathematical knowledge worthwhile.
2. Leaving behind something of permanent value is a noble ambition for anyone to undertake.
3. Mathematics is worth doing because its patterns are *beautiful*.
4. The best mathematics is not only beautiful, it is also *important*, as opposed to 'trivial' mathematics. In that sense, there is a seriousness to it, which lies in the *significance* of the ideas which it connects.
5. Seriousness of mathematical truth is characterized by its *generality* and its *depth*.
6. Good proofs are characterized by a high degree of surprise,[3] combined with *inevitability* and *economy*.
7. Mathematics is often considered to be important because it is *useful*. This (for Hardy) is a misconception. Pure mathematics is all about aesthetics, but mathematical *technique* is taught mainly through pure mathematics, and it is technique that is most useful.
8. Applied mathematics deals with reality, and hence is more useful than pure mathematics, but in some sense, applied mathematics is not 'real' mathematics (which deals only with abstract patterns).

There are many interesting notions here to reflect on: the character of mathematics, what constitutes good mathematics, and its usefulness.

[2] STEM stands for Science, Technology, Engineering and Medicine.
[3] Hardy uses the term 'unexpectedness'.

(i) Usefulness. This is what bothers Hardy most, and he builds elaborate arguments and draws surprising conclusions from them. For Hardy, pure mathematics is an *unprofitable, perfectly harmless and innocent occupation*. To him, it is 'unprofitable' in the sense of being unlike medicine where knowledge of the subject leads directly to making money. (In our society, with its craze for engineering and medicine, this is easy to understand.) It is 'harmless' in being not directly useful in warfare.

Poor Hardy was proved hopelessly wrong in just a few years after publication of the book. His beloved number theory was important in decoding the German Enigma machine during the Second World War. Worse, quantum mechanics (listed among 'useless' mathematics by Hardy in Section 26) was critically used in the making of the atomic bomb that caused unprecedented destruction in Hiroshima and Nagasaki. For a professed anti-war peace-activist like Hardy, this was heart-breaking.

In the century since, we have seen dramatic applications and use of what Hardy called 'pure' mathematics, in unanticipated areas. The theorems of Hardy and Ramanujan find newer applications to this day. So when Hardy says, *I have never done anything 'useful'*, he could not have been more wrong.

Therein lies a lesson for policy makers and governments who demand applications in 'thrust areas.' Mankind has been notoriously bad at predicting what knowledge will be useful in future.

There is another side to Hardy's distinctions between 'pure' and 'applied' mathematics. For Hardy, those aspects of mathematics that serve as tools for engineers and applied scientists are precisely the "boring" ones.

(ii) Character. Hardy is at his best when he explains his view of mathematics as a creative art. Perhaps the most famous quote from the essay is also one that beautifully expounds this view (section 10):

> *A mathematician, like a painter or a poet, is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made with* ***ideas****. . . . The mathematician's patterns, like the painter's or the poet's, must be* ***beautiful*** *; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test; there is no permanent place in the world for ugly mathematics.*

Hardy admits that it may be difficult to define mathematical beauty but that does not prevent us from recognizing such beauty when we see it. Where does beauty reside in mathematics? In the intricacy of connections, in the surprise that such connections reveal, and in the significance and depth of such revelations. The significance is not in terms of the consequences, but in terms of its impact on mathematics itself. In all this, there is an emphasis on technique, which is critical. Here is Hardy again (section 8):

> *(The mathematician's) subject is the most curious of all – there is none in which truth plays such odd pranks. It has the most elaborate and the most fascinating technique, and gives unrivaled openings for the display of sheer professional skill.*

The latter statement might well apply to music, but the former, in relating to truth- seeking, distinguishes mathematics. This principally aesthetic character of mathematics is the central thesis of Hardy's essay.

(iii) Seriousness. Where does beauty reside in mathematics? Is it only in intellectual pursuit of abstraction? That is not so, and Hardy cites the example of Chess. The tremendous difficulty in playing Chess also consists of intricate work on abstractions. But this is 'trivial' mathematics for Hardy. This is often misunderstood and dismissed as arrogance.[4] For Hardy, what distinguishes chess problems and mathematics is that the

[4] There are many discussions on Hardy's book on the Internet, and many take this attitude.

former is particular and the solution (winning) is an end in itself, while the latter is universal and every solution leads to new questions, extensions and generalisations. It is this inherently unending quest that helps the mathematician build layer upon layer of abstraction, which in turn characterises *depth* of mathematical thought.

Hardy presents two theorems and their proofs to illustrate such aesthetics. Both are familiar to children in high school: one is Euclid's demonstration that there are infinitely many primes, and the second is Pythagoras's proof that $\sqrt{2}$ is irrational. Both proofs demonstrate the element of surprise and the virtues of inevitability and economy of notions that Hardy considers essential for what constitutes beauty and elegance in mathematics.

Relevance

All this sounds very philosophical, so one might well ask: in what way is Hardy's apology relevant to the mathematics teacher of today? In my opinion, the relevance is direct and often unacknowledged.

Schooling is compulsory and mathematics is a compulsory subject. Yet, barring a few, every child asks: why should I learn mathematics? It is important to grant the child the right to ask this question, and acknowledge that the question is asked not in a bright and inquisitive mood but in one of immense frustration. In this context, there are some questions that we need answer honestly and sincerely.

- Why should one study **mathematics?**
- Why should **I** study mathematics?
- What can a (reasonably sincere) student expect to learn from ten years of school mathematics?

If the first is answered by pointing to mathematics being useful in everyday life, the truth is that, except in a few professions, everyday use of mathematics rarely goes beyond what is taught in elementary school.[5] If the answer is 'use in science' (and this is indeed critical), calculators are able to take on that role nowadays.

We usually go farther, point to the use of mathematics in all disciplines, its immense applicability in life, and so on. But that is where the second question above is relevant. Even if I were to grant that mathematics is important, I could wonder why I should learn it, since others who like it could very well take on that important work, leaving me to do unimportant things that I like. Importance and usefulness to society are no source of comfort.

For those who get past the first two, who consent to learn mathematics, the third is still relevant, since what one learns through ten years of schooling may yet not be the important mathematics that was used to justify learning it in the first place! In what way is factoring monomials and manipulating trigonometric identities to be seen as useful either for my everyday life or for advancing applications in the world?

What we need is *A Mathematics Educator's Apology*, and Hardy provides us with a part of it.

The National Curriculum Framework (NCF) [NCERT, 2005] speaks of the mathematization of the child's thought processes as the main goal of mathematics education. The position paper of NCF [PP 2005] goes on to cite George Polya and distinguish the 'narrow aims' and the 'higher aims' of mathematics education. The former relate to knowledge and skills that contribute to economic development, what Hardy would call 'useful' mathematics. The latter relate to the aesthetic dimension Hardy is at pains to elucidate.

In its vision, NCF wants children to learn *important mathematics*, and asserts: "Understanding when and how a mathematical technique is to be used is always more important

5 Sadly, the kind of mathematics that could be of immense use in most professions on an everyday basis like optimization and expectation are not taught in school at all.

than recalling the technique from memory." Once again, Hardy's distinction between 'real' mathematics and 'trivial' mathematics comes to mind.

The best answers we can give to children's questions above lie in viewing mathematics as a way of thinking, in the tremendous enrichment of "inner resources" it offers. This is not a luxury, but a process of realising one's potential. The aesthetic dimension of mathematics is often missed in school, and this impoverishes all of us. This dimension is reflected not in the content areas of mathematics, but in the wide range of processes at work in doing mathematics: formal problem solving, use of heuristics, estimation and approximation, optimization, use of patterns, visualization, representation, reasoning and proof, making connections, mathematical communication, etc. Indeed, according to the NCF:

> *Giving importance to these processes constitutes the difference between doing mathematics and swallowing mathematics, between mathematization of thinking and memorizing formulas, between trivial mathematics and important mathematics, between working towards the narrow aims and addressing the higher aims.*

Critical remarks

We cannot end this review without remarking on certain aspects of Hardy's style. There is much to offend in it too. It is a product of its time, and we cannot judge style across 70 years, but it is good to warn potential readers that it is implicitly sexist and highly Oxbridge-centric. There is a certain air of misanthropy ("Most people can do nothing at all well"), arrogance ("Is not the position of the ordinary applied mathematician a little pathetic?"), and contempt ("the intolerably ugly and incredibly dull school mathematics").

But Hardy does give us a real glimpse of what pure mathematics is all about, and ultimately, accomplishes what he sets out to do: a justification for doing mathematics.

Reference:

1. (NCF, PP): Position paper on teaching of mathematics by the National Focus Group, National Curriculum Framework – 2005, National Council for Educational Research and Training (NCERT), New Delhi 2006. http://www.ncert.nic.in/html/focus_group.htm.

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