Problem Corner

ADVENTURES IN PROBLEM SOLVING
Some Problems from the pre-RMO 2017

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In this edition of 'Adventures' we study a few problems from the pre-RMO, which was held in various centres across India, on 20 August 2017.

As usual, we pose the problems first and give the solutions later in the article, thereby giving you an opportunity to work on the problems.

(The problems are numbered as in the actual paper.)

Problems

Problem 8: A pen costs ₹ 11 and a notebook costs ₹ 13. Find the number of ways in which a person can spend exactly ₹ 1000 to buy pens and notebooks.

Problem 14: Suppose x is a positive real number such that $\{x\}$, $[x]$ and x are in geometric progression. Find the least positive integer n such that $x^n > 100$. (Here $[x]$ denotes the integer part of x and $\{x\} = x - [x]$.)

Problem 16: Five distinct 2-digit numbers are in geometric progression. Find the middle term.

Problem 20: What is the number of triples (a, b, c) of positive integers such that (i) $a < b < c < 10$ and (ii) $a, b, c, 10$ form the sides of a quadrilateral?

Problem 21: Find the number of ordered triples (a, b, c) of positive integers such that $abc = 108$.

Keywords: integers, exponents, real numbers, quadrilaterals, factors, primes, combinatorics

Solutions

Problem 8. *A pen costs ₹ 11 and a notebook costs ₹ 13. Find the number of ways in which a person can spend exactly ₹ 1000 to buy pens and notebooks.*

Solution. Let x be the number of pens and y the number of notebooks. Then we have $11x + 13y = 1000$. Hence we must find the number of nonnegative integer solutions to the equation $11x + 13y = 1000$. Note that this is a single equation with two unknowns. So its general solution must involve one free parameter.

To find the parametric equation, we need any one integer solution to the equation; we can find this by inspection or trial. Here is one such: $x = 20$, $y = 60$ (corresponding to $220 + 780 = 1000$). Next, we need a parametric integer solution to the homogeneous equation $11x + 13y = 0$ (note that the number on the right side is 0). An obvious solution is $x = 13m, y = -11m$, where m is any integer. This captures all integer solutions to the homogeneous equation. Combining these two solutions, we see that a general integer solution to the equation $11x + 13y = 1000$ is the following:

$$x = 20 + 13m, \quad y = 60 - 11m.$$

where m is an arbitrary integer. *Importantly, this parametric solution captures all integer solutions to the given equation.*

As our interest is only in nonnegative integer solutions, we must have

$$20 + 13m \geq 0, \quad 60 - 11m \geq 0.$$

The two inequalities yield:

$$m \geq -\frac{20}{13}, \quad m \leq \frac{60}{11},$$

i.e., $m \geq -1$ and $m \leq 5$, since m is an integer. The integer values allowed by these constraints are $-1, 0, 1, 2, 3, 4, 5$, or 7 values in all. Hence the required number of ways is 7.

Problem 14. *Suppose x is a positive real number such that $\{x\}$, $[x]$ and x are in geometric progression. Find the least positive integer n such that $x^n > 100$.*

Solution. Let $[x] = m$ and $x - [x] = t$; then $m \geq 0$, $0 \leq t < 1$, and $t, m, m + t$ are in GP. Note that this means that $m \neq 0$; a GP cannot have a 0 in it. Hence $m \geq 1$ and $x \geq 1$. Let r be the common ratio of the GP; then

$$r = \frac{m}{t}, \quad r = \frac{m+t}{m}.$$

Since $0 \leq t < 1$, we have

$$r > m, \quad r = 1 + \frac{t}{m} < 1 + \frac{1}{m}.$$

These relations imply that

$$m < 1 + \frac{1}{m}.$$

This inequality is true for $m = 1$ and is false for all $m \geq 2$. Hence $m = 1$. So the GP is $t, 1, 1 + t$. Hence we have $t(1 + t) = 1$, i.e., $t^2 + t - 1 = 0$, and so

$$t = \frac{-1 \pm \sqrt{5}}{2}.$$

The negative sign cannot apply since $t \geq 0$; hence $t = \frac{1}{2}(\sqrt{5} - 1)$ and

$$x = \frac{\sqrt{5}+1}{2} \approx 1.618.$$

So x is the famous Golden Ratio. Its defining property is the relation $x^2 = x + 1$. We now observe the following about the powers of x:

$$\begin{aligned}
x^2 &= 1x + 1, \\
x^3 &= x \cdot x^2 = x(x+1) = x^2 + x = 2x + 1, \\
x^4 &= x \cdot x^3 = x(2x+1) = 2x^2 + x = 3x + 2, \\
x^5 &= x \cdot x^4 = x(3x+2) = 3x^2 + 2x = 5x + 3,
\end{aligned}$$

and so on. From the last two relations we get:

$$\begin{aligned}
x^9 &= (3x+2)(5x+3) = 15x^2 + 19x + 6 \\
&= 15(x+1) + 19x + 6 = 34x + 21, \\
x^{10} &= (5x+3)^2 = 25x^2 + 30x + 9 \\
&= 25(x+1) + 30x + 9 = 55x + 34.
\end{aligned}$$

Note the striking pattern: *each power of x may be expressed in the form $ax + b$ where a and b are two consecutive Fibonacci numbers.* From the last two relations, we observe that:

$$x^9 \approx 34(1.62) + 21 < 100,$$

$$x^{10} \approx 55(1.62) + 34 > 100.$$

Hence the desired value of n is 10.

Problem 16. *Five distinct 2-digit numbers are in geometric progression. Find the middle term.*

Solution. There is no loss of generality in assuming that the numbers of the GP are in increasing order; if not, we simply reverse the order of the numbers (the GP remains a GP!). We may therefore assume that the common ratio of the GP exceeds 1. Let a be the first term of the GP, $a \geq 10$, and let $r = m/n > 1$ be the common ratio, where m, n are positive integers with no factors in common ($m > n$). The fifth term of the GP is $ar^4 \geq 10r^4$. Since the fifth term is a two-digit number, we have $10r^4 < 100$, hence $r < 2$. It follows that r is a rational number lying between 1 and 2, and therefore that $n \geq 2$ and $m \geq 3$. The terms of the GP are:

$$a, \quad \frac{am}{n}, \quad \frac{am^2}{n^2}, \quad \frac{am^3}{n^3}, \quad \frac{am^4}{n^4}.$$

As each of these quantities is an integer, and m^4 and n^4 have no factors in common, it follows that n^4 divides a. Hence $a/n^4 \geq 1$. Since $am^4/n^4 < 100$, it follows that $m^4 < 100$ and therefore that $m \leq 3$. Combining this with the fact that $m \geq 3$, we deduce that $m = 3$ and hence that $n = 2$; so $r = 3/2$. Therefore a is a multiple of 16. Focusing attention on the fifth term, we see that

$$\frac{a \times 3^4}{2^4} < 100, \qquad \therefore \ a < 100 \times \frac{16}{81} < 20.$$

Hence $a = 16$ and the GP is $16, 24, 36, 54, 81$. The middle term is 36.

Problem 20. *What is the number of triples (a, b, c) of positive integers such that (i) $a < b < c < 10$ and (ii) $a, b, c, 10$ form the sides of a quadrilateral?*

Solution. The triangle inequality shows that a quadrilateral with prescribed sides exists if and only if the sum of the lengths of the three shortest sides exceeds the longest side. Therefore we must find the number of triples (a, b, c) of positive integers such that $a < b < c < 10$ and $a + b + c > 10$. Since $a + b + c$ is at most equal to $c - 2 + c - 1 + c = 3c - 3$, we must have $3c - 3 > 10$, i.e., $c > 13/3$. Since c is an integer, this tells us that $c \geq 5$. Therefore the possible values that c can take are $5, 6, 7, 8, 9$. We consider each value in turn, starting from the top.

c = 9: We must have $a < b \leq 8$ and $a + b > 1$. The inequality will be satisfied for all possible choices of a, b, hence the number of choices is $\binom{8}{2} = 28$.

c = 8: We must have $a < b \leq 7$ and $a + b > 2$. The inequality will be satisfied for all possible choices of a, b, hence the number of choices is $\binom{7}{2} = 21$.

c = 7: We must have $a < b \leq 6$ and $a + b > 3$. The inequality will be satisfied for all choices of a, b except for $(a, b) = (1, 2)$, hence the number of choices is $\binom{6}{2} - 1 = 15 - 1 = 14$.

c = 6: We must have $a < b \leq 5$ and $a + b > 4$. The inequality will be satisfied for all choices of a, b except for $(a, b) = (1, 2)$ and $(1, 3)$, hence the number of choices is $\binom{5}{2} - 2 = 10 - 2 = 8$.

c = 5: We must have $a < b \leq 4$ and $a + b > 5$. The inequality will be satisfied only by $(a, b) = (3, 4)$ and $(2, 4)$, hence the number of choices is 2.

So the total number of choices available is $28 + 21 + 14 + 8 + 2 = 73$.

Problem 21. *Find the number of ordered triples (a, b, c) of positive integers such that $abc = 108$.*

Solution. Since the prime factorisation of 108 is $108 = 2^2 \times 3^3$, the only primes that divide a, b, c must be 2 and 3. We must 'distribute' two 2's and three 3's across a, b, c. The ways to distribute two 2's across a, b, c are the following:

- $2 + 0 + 0$, which has 3 permutations;
- $1 + 1 + 0$, which too has 3 permutations.

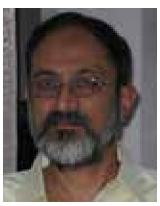
Hence there are $3 + 3 = 6$ ways of distributing two 2's across a, b, c.

The ways to distribute three 3's across a, b, c are the following:

- $3 + 0 + 0$, which has 3 permutations;
- $2 + 1 + 0$, which has 6 permutations;
- $1 + 1 + 1$, which has just 1 permutation.

Hence there are $3 + 6 + 1 = 10$ ways of distributing three 3's across a, b, c.

Any of the ways of distributing the twos can be combined with any of the ways of distributing the threes, hence the total number of ways is $6 \times 10 = 60$. This is the required answer.



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