# On the sums of powers of NATURAL NUMBERS Part 1

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## Introduction

The formulae to obtain the sum of the first $n$ natural numbers, the sum of the squares of the first $n$ natural numbers and the sum of the cubes of the first $n$ natural numbers are generally introduced to students during their final school year. Generally, these formulae are derived using the celebrated technique of mathematical induction. But the defect of this approach is that we must know in advance the formula to be proved. This is clearly a great handicap.

This article consists of two parts. In Part 1, we construct situations or 'stories'; in solving the problems posed in these narratives, we derive the different formulae. By relating the mathematics to real-life situations, teaching and learning become lively and enjoyable. It is hoped that this method will encourage mathematics teachers to create relevant stories to introduce some topics of mathematics, right from the early years of mathematics education.

In Part 2, we present a remarkable triangular arrangement of numbers — somewhat like the famous Pascal triangle — which helps in obtaining an expression for the sum of the $k$-th powers of the first $n$ natural numbers, for any given positive integer $k$.

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**Sum of the squares of the first n natural numbers**

**Problem 1.** Suppose a patient has to undergo, on the advice and prescription of an expert physician, three medical tests A, B and C (say a urine test, a blood test, and an X-ray, to be specific). The physician has imposed certain conditions on these tests:

(i) Tests B and C must be done only after test A is done.

(ii) Test A takes a full day; no other test can be carried out that day.

(iii) Each test can be completed on a single day; its implementation does not spill over to the next day.

(iv) Tests B and C can be carried out on the same day, or on separate days, in either order. When two tests are done together on a single day, we consider them to be a single test. The order in which the tests are performed is of no importance in this case.

(v) The tests must be completed within $n+1$ days numbered $1, 2, 3, \ldots, n+1$.

The question now is: In how many different ways can tests A, B, C be performed?

**Solution.** Test A cannot be done on day $n+1$, as then no day will be left for the other tests, in view of conditions (i) and (ii). Let test A be done on the $k$-th day ($k = 1, 2, \ldots, n$). Having made this choice, the days available for tests B and C are $k+1$, $k+2$, …, $n+1$, i.e., $n+1-k$ in number. So having fixed day $k$ for test A ($k = 1, 2, \ldots, n$), tests B and C can each be performed in $n+1-k$ different ways. As there is no restriction on tests B and C (in terms of which is done first), they can be done in $(n+1-k) \times (n+1-k) = (n+1-k)^2$ different ways for each value of $k$ (from $k = 1, 2, \ldots, n$). So the total number of different ways in which all the tests A, B, C can be carried out is equal to

$$\sum_{k=1}^{n}(n+1-k)^2 = n^2 + (n-1)^2 + \cdots + 2^2 + 1^2, \tag{1}$$

i.e., the sum of the squares of the first $n$ natural numbers.

Now let us look at the problem in a different way. We can choose either 2 days for the tests or 3 days (from the available $n+1$ days). Since 2 days can be chosen in $\binom{n+1}{2}$ different ways, this gives the number of ways of carrying out the tests in the event that we do the tests on 2 days. Similarly, 3 days can be chosen in $\binom{n+1}{3}$ different ways. Let the three days be $u, v, w$ where $u < v < w$. Tests A, B, C can be conducted on days $u, v, w$ **or** on days $u, w, v$ respectively. Thus each such selection of three days gives rise to **two** different ways in which the tests can be carried out. This implies that the number of different ways in which tests A, B, C can be conducted is

$$\binom{n+1}{2} + 2 \cdot \binom{n+1}{3}. \tag{2}$$

From (1) and (2), we see that we have obtained two different expressions for the number of ways in which we can conduct the tests. The two expressions must yield the same number, which means that

$$n^2 + (n-1)^2 + \cdots + 2^2 + 1^2 = \binom{n+1}{2} + 2 \cdot \binom{n+1}{3}. \tag{3}$$

By simplifying the expression on the right side, we get an expression for the sum of the squares of the first $n$ natural numbers:

$$1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = \frac{n(n+1)(2n+1)}{6}. \tag{4}$$

Thus by casting the problem in a relevant setting, we have derived the result.

The readers will, we hope, appreciate this derivation as very different from the 'standard derivation' based on mathematical induction, and will also see the merit of this new way of deriving the result.

### Sum of the cubes of the first n natural numbers

**Problem 2.** A devotee is going to stay for $n+1$ days in a dormitory attached to a temple. During these $n+1$ days, the devotee proposes to offer four different rituals in the shrine: A, B, C, and D. Taking into consideration the traditions associated with these offerings, some restrictions have been placed on the rituals:

(i) Ritual A must precede rituals B, C, and D.

(ii) No other ritual must be performed on the day that ritual A is done.

(iii) Any ritual can be completed on a single day; no ritual will spill over to the next day.

(iv) Rituals B, C, D can be performed together on a single day or on separate days; or two of them can be performed on one day and the remaining on some other day. When two or more rituals are done together on a single day, we consider them to be a single ritual. The order in which the rituals are performed is of no importance in this case.

(v) The rituals must be completed within $n+1$ days numbered $1, 2, 3, \ldots, n+1$.

The question now is: In how many different ways can rituals A, B, C, D be done?

**Solution.** The solution to Problem 2 is obtained by following arguments similar to those used in the solution to Problem 1.

Suppose that ritual A is done on the $k$-th day ($k = 1, 2, \ldots, n$). This fixed, there are now $n+1-k$ choices for the days on which *each* of the remaining rituals (B, C, D) can be done. Hence, the total number of different ways in which all the four rituals A, B C, D can be performed is equal to

$$\sum_{k=1}^{n} (n+1-k)^3 = 1^3 + 2^3 + \cdots + (n-1)^3 + n^3. \tag{5}$$

Now let us look at the problem in the following way.

(i) Out of the available $(n+1)$ days, two different days can be chosen in $\binom{n+1}{2}$ ways. Let the chosen days be $u, v$, with $u < v$. We now do ritual A on day $u$ and the remaining three rituals B, C, D on day $v$. (The order in which the rituals are done on that day is not of any consequence.)

(ii) Out of the available $(n+1)$ days, three different days (say $u, v, w$, in that order) can be chosen in $\binom{n+1}{3}$ ways. For this choice, ritual A is done on day $u$, and rituals B, C, D can be performed in any of the six ways depicted below.

| Day $v$ | B | C, D | C | B, D | D | B, C |
|---|---|---|---|---|---|---|
| Day $w$ | C, D | B | B, D | C | B, C | D |

(iii) Out of the available $(n+1)$ days, four different days (say $u, v, w, x$, in that order) can be chosen in $\binom{n+1}{4}$ ways. For each such choice, we do ritual A on day $u$ and B, C, D on days $v, w, x$. Assigning rituals B, C, D to days $v, w, x$ can be done in $3! = 6$ different ways.

So, collecting the arguments just made in (i), (ii) and (iii), the total number of different ways in which the rituals A, B, C, D can be done is equal to

$$\binom{n+1}{2} + 6 \cdot \binom{n+1}{3} + 6 \cdot \binom{n+1}{4}. \tag{6}$$

On simplification, we get

$$1^3 + 2^3 + \cdots + (n-1)^3 + n^3 = \frac{n^2(n+1)^2}{4}. \tag{7}$$

As earlier, we have derived the known formula by narrating a situation.

**Remark.** The reader should be able to work out for himself or herself that by composing similar ‘stories’ with larger numbers of ‘tests’ or ‘rituals,’ expressions can be found which give the sum of the 4-th powers or 5-th powers or 6-th powers (or still higher powers) of the first $n$ positive integers. (The stories may however get progressively more complicated!) We leave the challenge of composing these stories to the reader.

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