Teaching the TERNARY BASE Using a Card Trick

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Any sufficiently advanced technology is indistinguishable from magic.

— Arthur C. Clarke, "Profiles of the Future: An Inquiry Into the Limits of the Possible"

Magic has the ability to grab the attention of adults and children alike, whether it is a traditional trick of pulling rabbits out of a hat or a more sophisticated mind-reading card trick. We started a project at our school to excite children about mathematics using card tricks. We were pleasantly surprised to see such a positive response from children across grades.

In the following article, we describe how we used the 27-card trick to introduce ternary bases to students in Grade 9.

The Magic Trick

Grab a deck of cards, remove the jokers and shuffle the deck. Select any 27 cards from the deck. Fan them out in front of your class and ask a volunteer to choose one card at random, show it to the rest of the class (turn your back or close your eyes so that it is clear you have not seen the selected card) and put the card back into the deck. This card will henceforth be called the secret card. Invite the volunteer to shuffle the deck as many times to the satisfaction of the spectators that the secret card is lost in the deck. While taking the deck back from the volunteer you proclaim that in three steps you will reveal the secret card.

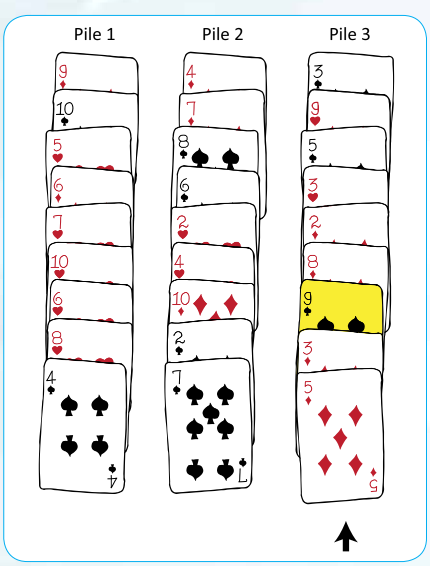

Figure 1

In our example, suppose the secret card is 9 of spades. We will use the notation 9S for 9 of spades, 9D for 9 of diamond, 9C for 9 of clubs and 9H for 9 of hearts. In Step I, deal out the cards face-up into 3 piles, each pile containing

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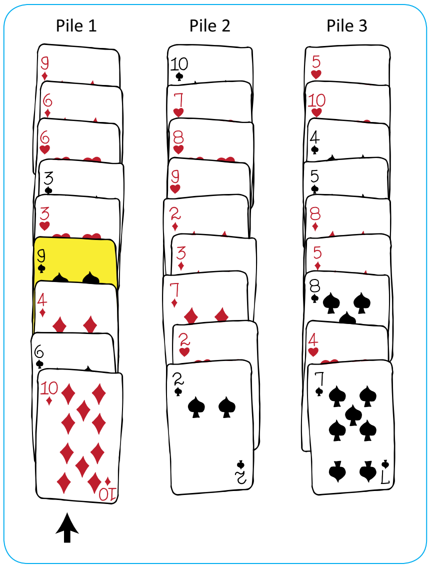

Figure 2

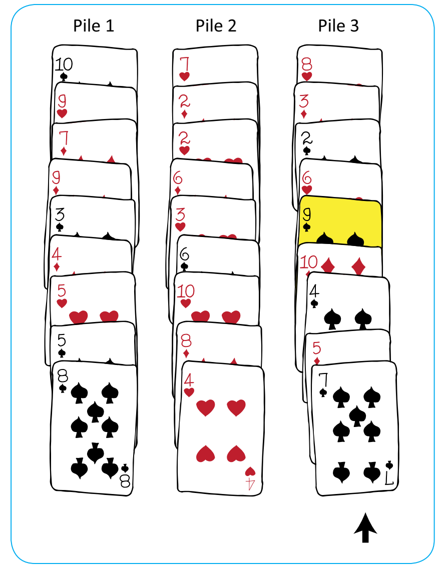

Figure 3

9 cards while the spectators keep watching. Referring to figure 1, the first card dealt is 9D, the second card is 4D, the third card is 3S, the fourth card is 10S and so on with card numbers 25, 26, 27 being the 4S, 7S and 5D respectively. Stack the cards in the first pile face-up, so that card #25 becomes the top card, and card #1 becomes the last card in the pile. Similarly, stack up the cards in the other two piles. For example, in the third pile, card #27 is the top card while card #3 is the last card. The volunteer now indicates the pile containing the secret card.

Turn the three piles over so that all the cards are face down. Pick up and stack the piles on your palm, one over the other, making sure to place the pile containing the secret card in-between the other two piles. In our example, since the secret card is in pile 3, this pile should be placed between pile 1 and 2. You may choose to place pile 2 on your palm first, follow it up with pile 3 and finish by placing pile 1 on the top. In other words, pile 2 is at the bottom, pile 3 at the middle and pile 1 is at the top.

In Steps II and III, we repeat the process of dealing the 27 cards face-up into 3 piles of 9 cards each, asking the volunteer to indicate the pile containing the secret card and placing this pile in-between the other two piles while collecting the cards. At the end of the three steps, we are ready to reveal the secret card.

Figure 2 shows the configuration of the cards when they are dealt in Step II. Since the secret card is now in pile 1, the piles should be collected and placed on the palm so that pile 1 is in between pile 2 and pile 3. Let us suppose you choose to place the piles such that pile 2 is at top, pile 1 in the middle and pile 3 at the bottom. Figure 3 shows the configuration of the cards when they are dealt out in Step III. Now the secret card is in pile 3. Therefore you must collect the piles in the order that ensures pile 3 is at the middle. Notice that at the end of Step III, the secret card will always be at position 14 from top (or bottom), which happens to be the middle card of the stack. This is clear from Figure 3.

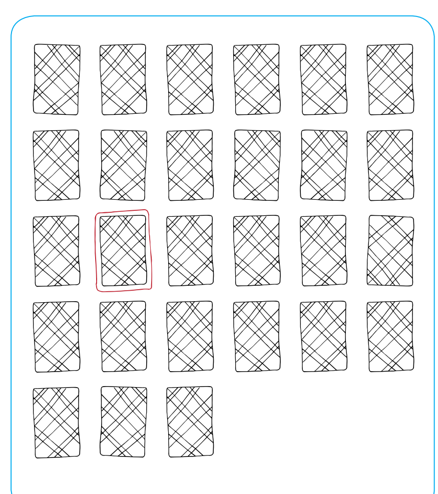

Figure 4

With the stack in your hand, all the cards face down deal out the cards (either face up or face down), placing them randomly on the table but carefully keeping track of the 14th card. After dealing out all the cards, pretend that you have lost track of the secret card, and at the moment the audience is about to acknowledge your defeat, point out the 14th card, proclaiming that this is indeed the secret card. Acknowledge and enjoy the applause from the audience!

Why does the trick work?

Why does the trick work? Denote by n the position of the secret card in the deck (counting from the top) after the volunteer has inserted it into the deck and the deck has been shuffled thoroughly. Thus $1 \leq n \leq 27$. For example, if $n = 15$, it means there are 14 cards on top of the secret card and 12 cards below it. Let us compute the new position of the secret card after Step I, i.e., after the cards have been dealt out and the cards collected back with the pile containing the secret card in-between the other two piles.

When the cards are dealt into three piles, the first, second and third cards in the deck become the first cards of piles 1, 2 and 3 respectively. Similarly, the fourth, fifth and sixth cards in the deck become the second cards in piles 1, 2 and 3 respectively, and so on. You can check for yourself that the 16th, 17th and 18th cards in the deck become the sixth cards of piles 1, 2 and 3 respectively.

From this we construct a mathematical function f which captures the position of the selected card in the deck after the 27 cards have been dealt into three piles of 9 each.

- If n is a multiple of 3, say $n = 3m$, then n goes into the third pile and assumes position m, i.e., $f(n) = m$. For example, if the initial position of the secret card is $n = 12$, then the secret card assumes position 4 in the third pile.
- Now consider the situation when n is not a multiple of 3. Suppose that $n = 20$. Note that 20 lies between 18 and 21 (these two numbers being consecutive multiples of 3 on either side of 20). With 18 cards already dealt into three piles containing 6 cards each, card #19 becomes the 7th card in pile 1, card #20 becomes the 7th card in pile 2, and card #21 becomes the 7th card in pile 3.

Note that

$$\frac{19}{3} = 6 + \frac{1}{3}, \qquad \frac{20}{3} = 6 + \frac{2}{3}, \qquad \frac{21}{3} = 7.$$

So we want f to be such that $f(19) = 7$, $f(20) = 7$ and $f(21) = 7$. From these observations, it is not difficult to see that

$$f(n) = \left\lceil \frac{n}{3} \right\rceil, \tag{1}$$

where $\lceil x \rceil$ is the *ceiling function*, defined thus: $\lceil x \rceil$ *is the smallest integer greater than or equal to x.* (Examples: $\lceil 1.7 \rceil = 2$, $\lceil \pi \rceil = 4$, $\lceil 8 \rceil = 8$.)

In the discussion below, we will make repeated use of the following important property of the ceiling function (try to find a proof of it for yourself):

Theorem. *For any two positive real numbers m and n,*

$$\left\lceil \frac{\lceil m \rceil}{n} \right\rceil = \left\lceil \frac{m}{n} \right\rceil.$$

Position of the selected card after Step I. We deduce the following from the considerations above: if the selected card is at position n_0 at the start, then after Step I, when a pile of 9 cards is placed over the pile containing the secret card, it is at position n_1, where

$$n_1 = 9 + \left\lceil \frac{n_0}{3} \right\rceil. \tag{2}$$

We call this operation M, with M signifying that the pile containing the secret card is placed in the 'middle position'. For example, if the secret card was at position $n_0 = 23$ initially, then after Step I the card will be at position n_1 where

$$n_1 = M(23) = 9 + \left\lceil \frac{23}{3} \right\rceil = 17.$$

At the end of Step II the card will be at position n_2 from the top, where

$$\begin{aligned} n_2 = M(M(n_0)) &= 9 + \left\lceil \frac{n_1}{3} \right\rceil \\ &= 9 + \left\lceil \frac{17}{3} \right\rceil = 9 + 6 = 15. \end{aligned}$$

So the 23rd card will land up in the 15th position. At the end of Step III the position of the card is n_3 where

$$n_3 = M(M(M(n_0))) = 9 + \left\lceil \frac{n_2}{3} \right\rceil$$
$$= 9 + \left\lceil \frac{15}{3} \right\rceil = 14.$$

For the trick to work, it must happen that for any n_0 with $1 \le n_0 \le 27$,

$$M(M(M(n_0))) = 14. \qquad (3)$$

To show this, note that

$$n_3 = 9 + \left\lceil \frac{n_2}{3} \right\rceil,$$
$$n_2 = 9 + \left\lceil \frac{n_1}{3} \right\rceil = \left\lceil \frac{27+n_1}{3} \right\rceil,$$
$$\therefore\ n_3 = 9 + \left\lceil \frac{1}{3} \left\lceil \frac{27+n_1}{3} \right\rceil \right\rceil$$
$$= 9 + \left\lceil \frac{27+n_1}{9} \right\rceil = 12 + \left\lceil \frac{n_1}{9} \right\rceil.$$

Next,

$$n_1 = 9 + \left\lceil \frac{n_0}{3} \right\rceil = \left\lceil \frac{27+n_0}{3} \right\rceil,$$
$$\therefore\ \left\lceil \frac{n_1}{9} \right\rceil = \left\lceil \frac{1}{9} \left\lceil \frac{27+n_0}{3} \right\rceil \right\rceil = \left\lceil \frac{27+n_0}{27} \right\rceil$$
$$= 1 + \left\lceil \frac{n_0}{27} \right\rceil = 1 + 1 = 2,$$

since $\left\lceil \frac{n_0}{27} \right\rceil = 1$ (because $1 \le n_0 \le 27$). This leads to:

$$n_3 = M(M(M(n_0))) = 13 + \left\lceil \frac{n_0}{27} \right\rceil = 14. \qquad (4)$$

It is to be noted that 14 is the middle position in a deck of 27 cards. The trick would also have worked if we had used 21 cards and dealt the cards into 3 piles, each with 7 cards and repeated Steps I, II, III as described above. After three steps, the selected card will be at position 11, the middle position in a deck of 21 cards; see [1].

Generalization of the Trick

We now modify the above magic trick in a way that leads to the development of the ternary base. (See Box 1 for an explanation of what is meant by *ternary base*.) The question is this: *Rather than bring the selected card to position* 14 *(i.e., the middle of the pack), can we perform the steps in such a way as to bring the selected card to some other desired position?* Let's try out an example.

As earlier, fan out the 27 cards face-down and ask a volunteer to select a card, show it to the audience and put it back in the deck. All this while you turn your back so that it is clear that you have not seen the secret card. Now ask the volunteer to select a number n between 1 and 27. While n is being chosen, shuffle the deck to the satisfaction of the audience. Unlike the previous trick, where in three steps the card is positioned at the centre of the deck, now the task is to place the card at position n by repeating three times the process of distributing the cards into 3 piles of 9 cards each and picking them up in some order. Naturally, the order in which you collect the cards at each step is crucial for the trick to work.

We will illustrate this with three examples.

Example 1. Let $n = 23$. This means that at the end of Step III, we want the secret card to be at position 23.

Recall that Step III consists of laying out the cards in three piles, asking the volunteer to indicate the pile containing the secret card followed by picking up the three piles. Observe that there are three ways in which the piles can be collected:

T: Put the pile containing the secret card on **Top** of the other two piles.

M: Put the pile containing the secret card in the **Middle**, between the other two piles.

B: Put the pile containing the secret card at the **Bottom** of the other two piles.

If the secret card is to be placed in the 23rd position, we must have 22 cards on top of this card. Since each pile contains 9 cards, we collect the two piles which do not contain the secret card (a total of 18 cards) and place them over the pile containing the secret card. This is procedure **(B)**. Obviously, this is not enough to ensure that the secret card is in position 23. For example, suppose

Ternary base

Ternary base is simply 'base 3'. Most of you will be familiar with base 2, also called the **binary system**, in which the only digits used are 0 and 1. In this system, we express the positive integers as sums of distinct powers of 2. For example:

$$6 = 2^2 + 2^1 = (110)_{\text{base two}},$$

$$7 = 2^2 + 2^1 + 2^0 = (111)_{\text{base two}},$$

$$11 = 2^3 + 2^1 + 2^0 = (1011)_{\text{base two}},$$

$$19 = 2^4 + 2^1 + 2^0 = (10011)_{\text{base two}},$$

and so on. The system works because every positive integer can be expressed in a **unique** manner as a sum of distinct powers of 2; the word 'unique' signifies that there is just one way of writing the sum. Each power of 2 is either present in the sum or not present, and this leads to the digits being either 1 or 0.

In the same way, we can express every positive integer in terms of the powers of 3. However, here we cannot say 'sum of distinct powers of 3'; for example, we cannot write 6 as a sum of distinct powers of 3. But we can express every positive integer as a sum of powers of 3 provided that *we permit each power to be not used, or used once, or used twice*; further choices are not needed. For example:

$$4 = 3^1 + 3^0 = (11)_{\text{base three}},$$

$$5 = 3^1 + 2 \cdot 3^0 = (12)_{\text{base three}},$$

$$6 = 2 \cdot 3^1 = (20)_{\text{base three}},$$

$$7 = 2 \cdot 3^1 + 3^0 = (21)_{\text{base three}},$$

$$8 = 2 \cdot 3^1 + 2 \cdot 3^0 = (22)_{\text{base three}},$$

and so on. Each of the above expressions is unique.

the secret card is at position 7 in pile 1. Then if we put piles 2 and 3 over pile 1 and deal out the cards, the secret card will be at position $18 + 7 = 25$. To be at position 23, we must ensure that the secret card is at position 5 *in its own pile* after the cards are dealt out in Step III.

How do we do this? When we deal out the 27 cards at the start of Step III, the 5th positions in piles 1, 2 and 3 will be filled by the 13th, 14th and 15th cards in the deck respectively. So at the end of Step II, when we have collected the 3 piles, the secret card must be either the 13th, 14th or 15th card in the deck. Thus our problem is reduced to placing the secret card at positions 13, 14 or 15 in the deck at the end of Step II. This requires that we follow procedure **(M)** in Step II: take the pile of 9 cards containing the secret card, put one pile on top of it and another pile below it. Is this enough to ensure that the secret card is at positions 13, 14 or 15 at the end of Step II? The answer is Yes only if the secret card is in the 4th, 5th or 6th position in its own pile when the cards are dealt out at the start of Step II.

Note that when a deck of 27 cards is dealt out, the cards in positions 10 to 18 will always take positions 4, 5 or 6 in their respective piles. For example, the 11th card will be placed at position 4 in the second pile, the 15th card will be placed at position 5 in the third pile, and the 16th card will be placed at position 6 in the first pile. Therefore the problem of placing the secret card in the 4th, 5th or 6th rank in its pile at the beginning of Step

II is now further reduced to placing the secret card in any of the positions 10 to 18 in the deck at the end of Step I. Which of the operations **(T)**, **(M)**, **(B)** of collecting the cards at the end of Step I will ensure this? The answer is operation **(M)**, so that there are 9 cards on top of the pile containing the secret card, and the secret card will thus be placed in one of the positions 10–18 in the deck.

To summarize, we perform operations **(B)**, **(M)**, **(M)** in Steps III, II and I respectively to bring the secret card at position 23. Let us put some numbers on these operations, say, **(T)** $= 0$, **(M)** $= 1$, **(B)** $= 2$. The idea behind these numbers is that with operations **(T)**, **(M)** and **(B)**, we are putting 0, 1 and 2 piles respectively on top of the pile containing the secret card. Note that $23 - 1 = 22 = 2 \times 3^2 + 1 \times 3^1 + 1 \times 3^0$, which shows that the number of cards on top of the secret card, 22 in our example, can be constructed by using the operations in the sequence **(M)**, **(M)**, **(B)**. This is nothing but the ternary base representation of 22, in disguise.

Example 2. Let $n = 21$. We wish to find the operations that will place the secret card in position 21.

As $21 = 18 + 3$, the third operation has to be **(B)** so that there are 18 cards on top of the pile containing the secret card. Moreover, the secret card must be in the 3rd position *in its own pile* at the end of Step III. This means that at the beginning of Step III the secret card has to be in the 7th, 8th or 9th position in the deck. How do we arrange for this configuration? This is only possible if in Step II we collect the cards by putting two piles of 18 cards below the pile containing the secret card. That is, we apply operation **(T)** in Step II. In addition to this, when we deal out the cards in Step II, the secret card must be either the 7th, 8th or 9th card in its own pile. Note that when a deck of 27 cards is dealt out, the cards ranked 19 to 27 will take positions 7, 8 or 9 *in their own piles*. For example, the 19th card will be placed at position 7 in the first pile, the 23rd card will be placed at position 8 in the second pile, and the 27th card will be placed at position 9 in the third pile. Therefore the problem of placing the secret card in the 7th, 8th or 9th rank *in its own pile* when the cards are dealt out in Step II is reduced to placing the secret card in the any of the positions 19 to 27 in the deck at the end of Step I. Hence in Step I we should collect the cards by applying the operation **(B)**, that is, put two piles of cards on top of the pile containing the secret card.

To summarize, we perform the operations **(B)**, **(T)**, **(B)** in Steps III, II and I respectively to bring the secret card at position 21. As **(T)** $= 0$, **(M)** $= 1$, **(B)** $= 2$ and $21 - 1 = 20 = 2 \times 3^2 + 0 \times 3^1 + 2 \times 3^0$, this shows that the number of cards on top of the secret card, 20 in our example, can be constructed by using the operations in the sequence **(B)**, **(T)**, **(B)**. This is again the representation of 20 in the ternary base.

Example 3. Let $n = 16$. We wish to find the operations that will place the secret card in position 16.

As $16 = 9 + 7$, the third operation has to be **(M)** so that there are 9 cards on top of the pile containing the secret card. Moreover, the secret card must be in the 7th position *in its own pile* in Step III. Therefore when we collect the cards at end of Step II, we have to put two piles of cards on top of the pile containing the secret card, which is operation **(B)**. In addition, we have to ensure that the secret card will be in the 1st, 2nd or 3rd position in its own pile when the deck is dealt out in Step II. Similar to the argument used in the above examples, note that when a deck of 27 cards is dealt out, the cards ranked 1 to 9 will take the positions 1, 2 or 3 *in their own piles*. Hence in Step I we should collect the cards by applying operation **(T)**.

Thus we are performing the operations **(M)**, **(B)**, **(T)** in Steps III, II and I respectively to bring the secret card to position 16. As **(T)** $= 0$, **(M)** $= 1$, **(B)** $= 2$ and

$$16 - 1 = 15 = 1 \times 3^2 + 2 \times 3^1 + 0 \times 3^0, \quad (5)$$

we see that the number of cards on top of the secret card, 15 in our example, can be constructed by using the operations in the sequence **(T)**, **(B)**, **(M)**. This is the ternary base representation of 15.

Mathematical Analysis

To find out how to place the card in the chosen position, let us revisit Step I in the previous trick. Suppose the secret card is at position n_0 from the top, after the card has been put back in the deck and the deck thoroughly shuffled. This means there are $n_0 - 1$ cards above it. You deal the 27 cards into 3 piles of 9 card each, in the manner described earlier, and ask the volunteer to indicate the pile which holds the selected card. At this point, the secret card is at position $\left\lceil \frac{n_0}{3} \right\rceil$ in the pile. Recall that there are now three ways in which you can collect the piles:

T: Put the pile containing the secret card on **Top** of the other two piles.

M: Put the pile containing the secret card in the **Middle**, between the other two piles.

B: Put the pile containing the secret card at the **Bottom** of the other two piles.

If the operation is **(T)**, then the position of the secret card in the deck remains unchanged at $\left\lceil \frac{n_0}{3} \right\rceil$. If the operation is **(M)**, the new position is $9 + \left\lceil \frac{n_0}{3} \right\rceil$, as 9 cards are put over the pile containing the secret card. If the operation is **(B)**, the secret card will be at the position $18 + \left\lceil \frac{n_0}{3} \right\rceil$, as 18 cards are put over the pile containing the secret card.

Let us denote the operation by F_a so that

$$F_a(n_0) = 9a + \left\lceil \frac{n_0}{3} \right\rceil, \tag{6}$$

where $a = 0$, 1 or 2. Note that $a = 0$ when the operation is **(T)**, $a = 1$ when the operation is **(M)** and $a = 2$ when the operation is **(B)**. Hence the trick will work if, given the chosen number n, we can succeed in finding three numbers $a, b, c \in \{0, 1, 2\}$ such that for any number n_0 between 1 and 27, we have

$$F_c(F_b(F_a(n_0))) = n. \tag{7}$$

The values of a, b, c will decide whether we have to perform operations **(T)**, **(M)** or **(B)** when collecting the cards in each of the three steps.

Now let us see what exactly is $F_c(F_b(F_a(n_0)))$. We have:

$$F_a(n_0) = 9a + \left\lceil \frac{n_0}{3} \right\rceil,$$

$$\therefore\ F_b(F_a(n_0)) = 9b + \left\lceil \frac{1}{3}\left(9a + \left\lceil \frac{n_0}{3} \right\rceil\right) \right\rceil.$$

Now,

$$\begin{aligned} \frac{1}{3}\left(9a + \left\lceil \frac{n_0}{3} \right\rceil\right) &= \left\lceil \frac{1}{3}\left\lceil 9a + \frac{n_0}{3} \right\rceil \right\rceil \\ &= 3a + \left\lceil \frac{n_0}{3^2} \right\rceil, \end{aligned}$$

implying that

$$F_b(F_a(n_0)) = 9b + 3a + \left\lceil \frac{n_0}{3^2} \right\rceil. \tag{8}$$

Using the same reasoning, it follows that

$$\begin{aligned} F_c(F_b(F_a(n_0))) &= 9c + 3b + a + \left\lceil \frac{n_0}{3^3} \right\rceil \\ &= 9c + 3b + a + 1, \end{aligned} \tag{9}$$

since $0 < \frac{n_0}{27} \le 1$ and hence $\left\lceil \frac{n_0}{3^3} \right\rceil = 1$.

Coming back to the trick, the analysis above demonstrates that if we want the secret card to be placed at position n in the deck of 27 cards, we must find numbers $a, b, c \in \{0, 1, 2\}$ such that $n = 9c + 3b + a + 1$. A shorthand way of writing this is

$$n = cba_{\text{base three}} + 1, \tag{10}$$

where the subscript 'base three' denotes that we are expressing the number n in the ternary base. For example, if the chosen number is $n = 17$, then we have to find a, b, c such that $9c + 3b + a = 16$. If we divide 16 by 9, we get 1 as the quotient and 7 as the remainder. Therefore $c = 1$. If we divide 7 by 3, we get 2 as the quotient and 1 as the remainder. Therefore $b = 2$ and $a = 1$. Thus $16 = 121_{\text{base three}}$ and after each dealing of the cards, they should be collected using the operations **(M)**, **(B)**, **(M)** respectively. This would make the secret card land up in the 17th position. As another example, if we want the secret card to be in the position 25, then we have to express 24 in the ternary base as $24 = 2 \times 9 + 2 \times 3 + 0 \times 1$ so that $a = 0, b = 2, c = 2$. Hence the operations of gathering the cards after they have been dealt out must be in the order **(T)**, **(B)**, **(B)** respectively.

Conclusion

The 27-card trick is a fascinating way to introduce the ternary base system to students. It intrigued the students to find out how the trick works. The same trick was shown repeatedly to the students until they saw a pattern emerging. We have been successful in making the students interested in learning about the ternary base through this magic.

References

1. https://www.youtube.com/watch?v=gcgvFTfOpD8
2. Gardner, Martin. *Mathematics, Magic and Mystery.* Dover Publications 1956.

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SOLUTIONS
NUMBER CROSSWORD

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	1: 2	2: 9				3: 3	4: 8	
	5: 1	1	6: 2		7: 9	7	6	
8: 2	1		9: 7	2	9		10: 1	4
	11: 7	12: 6	3		13: 9	14: 1	8	
		0				7		
	15: 2	6	16: 6		17: 3	5	18: 6	
19: 6	5		20: 9	2	5		21: 4	0
	22: 1	23: 8	3		24: 7	25: 3	0	
	26: 4	2				27: 7	3	

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