Student Corner – Featuring Articles Written by Students

An Approximation of π by using THE LAW OF COSINES

BODHIDEEP JOARDAR

The approximation of π is a popular pastime of students of mathematics and I have started with the familiar age-old way, of fitting a regular polygon tightly in a circle of radius r. The vertices of the n-sided polygon are joined to the centre of the circle so that the angle subtended at the centre by each segment is $360°/n$. If n is sufficiently large, the perimeter of the polygon approaches the circumference $2\pi r$ of the circle and this approximation improves as the number of segments increases. My object is to find a formula for π that is independent of the radius r.

The change that I have made is that I have used the cosine rule and a few trigonometric identities to calculate the perimeter of the polygon. Once I got a formula for π, my approximation has been improved with the use of a scientific calculator.

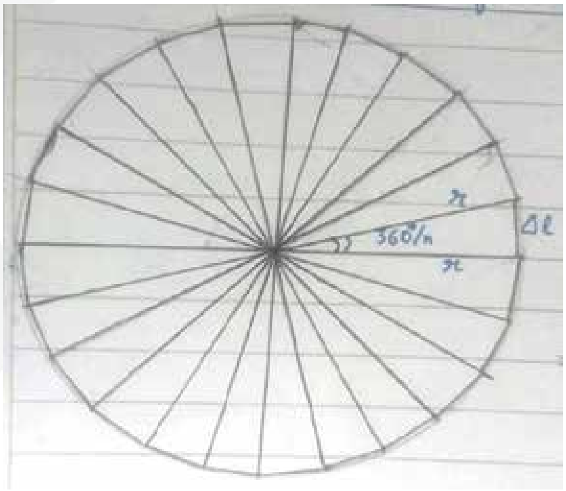

Figure 1. Up to this it is similar to Archimedes' construction, but instead of using Pythagoras' Theorem as he did, I want to use the law of cosines.

Keywords: π, approximation, circle, polygon, circumference, law of cosines, difference of cosines, limits

In Figure 1, the polygon is divided into n isosceles triangles, having equal arms of length r (the radius of the circle) and congruent to one another. Using the *law of cosines* the base $\triangle l$ of each triangle satisfies the relation,

$$(\triangle l)^2 = r^2 + r^2 - 2.r.r.\cos\frac{360^\circ}{n} = 2r^2\left(1 - \cos\frac{360^\circ}{n}\right)$$

Therefore, $\triangle l = r\sqrt{2\left(1 - \cos\frac{360^\circ}{n}\right)}$

For sufficiently large n, the perimeter of the polygon, i.e., $n.\triangle l$ will approximate the circumference of the circle.

So, the circumference $2\pi r \approx n.\triangle l = nr\sqrt{2\left(1 - \cos\frac{360^\circ}{n}\right)}$

Therefore, $\pi \approx \frac{n}{2}\sqrt{2\left(1 - \cos\frac{360^\circ}{n}\right)}$, which is independent of the radius r as it should be. *This is the desired formula for* π, which can be simplified further by using the identity

$$\cos A - \cos B = 2\sin\frac{B+A}{2}.\sin\frac{B-A}{2}.$$

So,

$$\pi \approx \frac{n}{2}\sqrt{2\left(1 - \cos\frac{360^\circ}{n}\right)} \approx \frac{n}{2}\sqrt{2\left(\cos 0^\circ - \cos\frac{360^\circ}{n}\right)}$$
$$\approx \frac{n}{2}\sqrt{2\left(2\sin\frac{180^\circ}{n}.\sin\frac{180^\circ}{n}\right)}$$

That is, $\pi \approx n\sin\frac{180^\circ}{n}$, *a simple formula.*

Taking $n = 24$, the value of π is 3.132628613281238…..

Taking higher and higher values of n and applying this formula, we can get better and better approximations of π. Let us look at Table 1.

n	π
24	3.132628613281238…
96	3.14103195089051…
576	3.141577077703974…
20736	3.141592641571345…
124416	3.141592653255947…
746496	3.141592653580528…

Table 1. Improved approximations for π

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Maryam Mirzakhani: A Tribute

Maryam Mirzakhani has gone the way of all too many great mathematicians who died young: Evariste Galois (1811–1832), Niels Abel (1802–1829), Srinivasa Ramanujan (1887–1920), Bernard Riemann (1826–1866), Gotthold Eisenstein (1823–1852), Frank Ramsey (1903–1930),... The list is already much too long, and now Maryam Mirzakhani (1977–2017) – to date, the only woman and the only mathematician from Iran to win a Fields medal (the top medal in mathematics and often referred to as the "Nobel prize of mathematics") – has joined it. She died in July 2017 of breast cancer, which she had had since 2013. She was 40 years old at the time of her death.

She won the Fields medal in 2014. The official citation read in part: [for] "... her outstanding contributions to the dynamics and geometry of Riemann surfaces and their moduli spaces."

She was a professor in Stanford University from 2009 till the time of her death. We give below a few extracts from the obituary which appeared in the Stanford newsletter [1]:

"Maryam is gone far too soon, but her impact will live on for the thousands of women she inspired to pursue math and science."

"Maryam was a brilliant mathematical theorist, and also a humble person who accepted honours only with the hope that it might encourage others to follow her path. Her contributions as both a scholar and a role model are significant and enduring, and she will be dearly missed."

She was a self-professed 'slow' mathematician. Her colleagues describe her as "resolute and fearless in the face of problems others would not, or could not, tackle. She denied herself the easy path, choosing instead to tackle thornier issues. Her preferred method of working on a problem was to doodle on large sheets of white paper, scribbling formulas on the periphery of her drawings. Her young daughter described her mother at work as 'painting.'"

"You have to spend some energy and effort to see the beauty of math," she told one reporter. In another interview, she said: "I don't have any particular recipe [for developing new proofs]... It is like being lost in a jungle and trying to use all the knowledge that you can gather to come up with some new tricks, and with some luck you might find a way out."

Reference

[1] http://news.stanford.edu/2017/07/15/maryam-mirzakhani-stanford-mathematician-and-fields-medal-winner-dies/