On Scalene Triples

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Given three positive integers a, b, c such that $a \geq b \geq c$, we say that the triple (a, b, c) has the **triangular property** if the sum of any two of the numbers exceeds the third one (i.e., $a + b > c$ and $b + c > a$ and $c + a > b$). The terminology draws from the well-known property of any triangle: the sum of any two of its sides exceeds the third side. We say further that the triangular triple (a, b, c) is **scalene** if a, b, c are distinct positive integers satisfying the triangular property. For example, the triple $(4, 3, 2)$ has the triangular property and it is scalene. However, the triples $(3, 2, 1)$ and $(5, 3, 1)$ do not have the triangular property.

Fix an upper limit n, and let a, b, c take all possible positive integer values from 1 to n (i.e., $1 \leq a, b, c \leq n$). Clearly, n^3 triples are possible. In this article, we study the following question:

> *Given a fixed positive integer n, how many scalene triples exist?*

Notation.

- $S(n)$ denotes the set of all triples (a, b, c) with $1 \leq a, b, c \leq n$. This may be regarded as the "universal set" within which the set we are studying resides (the triples satisfying the triangular property and ordered so that $a \geq b \geq c$). The number of triples in $S(n)$ is clearly n^3, i.e., $|S(n)| = n^3$.
- $T(n)$ denotes the set of triples in $S(n)$ which possess the triangular property.
- $\triangle(n)$ denotes the number of triples in $T(n)$ which are scalene, i.e., $\triangle(n)$ counts the scalene triples in $T(n)$.
- $s(n)$ denotes the number of scalene triples added when a positive integer n is introduced to the range. That is, $s(n) = \triangle(n) - \triangle(n-1)$ for $n > 1$. Stated otherwise, $\triangle(n) = s(1) + s(2) + s(3) + \cdots + s(n)$.

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Counting the number of scalene triples
To calculate the number of scalene triples formed by integers $1, 2, 3, \ldots, n$, we first enumerate the values of $\triangle(n)$ for a few small values of n.

Since we need three distinct positive integers to form a scalene triangle, it follows that $\triangle(1) = 0$ and $\triangle(2) = 0$. Further, for $n = 3$, the triple $(3, 2, 1)$ does not satisfy the triangular property; hence $\triangle(3) = 0$ too. The values of $\triangle(n)$ for $n = 4, 5, 6, 7, 8$ are computed as shown below, by actually listing all the triples which satisfy the stated property.

n	Scalene triples added	Number of triples added, i.e., $s(n)$	$\triangle(n)$
4	$(4,3,2)$	1	1
5	$(5,4,3)$, $(5,4,2)$	2	3
6	$(6,5,4)$, $(6,5,3)$, $(6,5,2)$ $(6,4,3)$	4	7
7	$(7,6,5)$, $(7,6,4)$, $(7,6,3)$, $(7,6,2)$ $(7,5,4)$, $(7,5,3)$	6	13
8	$(8,7,6)$, $(8,7,5)$, $(8,7,4)$, $(8,7,3)$, $(8,7,2)$ $(8,6,5)$, $(8,6,4)$, $(8,6,3)$ $(8,5,4)$	9	22

Counting the scalene triples for larger n. For larger n, let us list the possible scalene triangular triples (a, b, c) in some systematic manner. Remember that we need $c < b < a \leq n$ and $b + c > a$.

Suppose that $a = 2k$, where k is a positive integer. Since the entries must be unequal, we must have $b < 2k$; so the largest possible value that b can take is $2k - 1$. What about the least possible value? Since $b + c > a$ and $c < b$, we must have $b \geq k + 1$. (If $b = k$, then $b + c \leq 2k - 1 < 2k$, which violates the triangle inequality.) So if $a = 2k$, then b can take the values $2k - 1$, $2k - 2$, …, $k + 1$. For each of these b-values, we list the possible c-values which fit the requirement. For example, if $b = 2k - 1$, then c can take values from 2 till $2k - 2$, or $2k - 3$ values in all. Similarly, if $b = 2k - 2$, then c can take values from 3 till $2k - 3$, or $2k - 5$ values in all. In this manner, we construct the following table of possibilities corresponding to $a = 2k$:

$$(2k, 2k-1, 2k-2),\ (2k, 2k-1, 2k-3),\ (2k, 2k-1, 2k-4),\ \ldots,\ (2k, 2k-1, 2),$$
$$(2k, 2k-2, 2k-3),\ (2k, 2k-2, 2k-4),\ (2k, 2k-2, 2k-5),\ \ldots,\ (2k, 2k-2, 3),$$
$$\vdots$$
$$(2k, k+2, k+1),\ (2k, k+2, k),\ (2k, k+2, k-1),$$
$$(2k, k+1, k).$$

It follows that the total number of triples in this case is

$$s(2k) = (2k-3) + (2k-5) + \cdots + 5 + 3 + 1 = (k-1)^2. \tag{1}$$

(To see why, recall the formula for the sum of an arithmetic progression; or recall that the sum of the first n odd positive integers is equal to n^2.) By reasoning in exactly the same manner, we construct the following table of possibilities corresponding to $a = 2k+1$:

$$\begin{aligned}
&(2k+1, 2k, 2k-1),\ (2k+1, 2k, 2k-2),\ (2k+1, 2k, 2k-3),\ \ldots,\ (2k+1, 2k, 2),\\
&(2k+1, 2k-1, 2k-2),\ (2k+1, 2k-1, 2k-3),\ \ldots,\ (2k+1, 2k-1, 3),\\
&\vdots\\
&(2k+1, k+3, k+2),\ (2k+1, k+3, k+1),\ (2k+1, k+3, k),\ (2k+1, k+3, k-1),\\
&(2k+1, k+2, k+1),\ (2k+1, k+2, k).
\end{aligned}$$

It follows that the total number of triples added is

$$s(2k+1) = (2k-2) + (2k-4) + \cdots + 6 + 4 + 2 = k^2 - k. \tag{2}$$

(As earlier, we may use the formula for the sum of an arithmetic progression; or we may note that the required sum is twice the sum of the first $k-1$ positive integers.)

Theorem. *For fixed n, the number of scalene triples formed by positive integers $1, 2, \ldots, n$ is*

$$\triangle(n) = \begin{cases} \dfrac{k(k-1)(4k-5)}{6}, & \text{if } n = 2k,\\[2ex] \dfrac{k(k-1)(4k+1)}{6}, & \text{if } n = 2k+1. \end{cases} \tag{3}$$

Proof. For an even positive integer $n = 2k$,

$$\begin{aligned}
\triangle(n) &= s(1) + s(2) + \cdots + s(2k-1) + s(2k)\\
&= \sum_{i=1}^{k-1}(i^2 - i) + \sum_{i=1}^{k}(i^2 - 2i + 1)\\
&= \sum_{i=1}^{k-1}(2i^2 - 3i + 1) + k^2 - 2k + 1\\
&= 2 \cdot \frac{(k-1)k(2k-1)}{6} - 3 \cdot \frac{(k-1)k}{2} + k - 1 + k^2 - 2k + 1\\
&= \frac{k(k-1)(4k-5)}{6}, \quad \text{on simplification.}
\end{aligned}$$

Similarly, for an odd positive integer $n = 2k + 1$,

$$\begin{aligned}
\triangle(n) &= s(1) + s(2) + \cdots + s(2k-1) + s(2k) + s(2k+1) \\
&= \sum_{i=1}^{k}(i^2 - i) + \sum_{i=1}^{k}(i^2 - 2i + 1) \\
&= \sum_{i=1}^{k}(2i^2 - 3i + 1) \\
&= 2 \cdot \frac{k(k+1)(2k+1)}{6} - 3 \cdot \frac{k(k+1)}{2} + k \\
&= \frac{k(k-1)(4k+1)}{6}, \quad \text{on simplification.}
\end{aligned}$$

Hence the stated result.

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