TRIANGULAR NUMBERS

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Here is a popular story about the famous German mathematician Carl Friedrich Gauss (1777–1855). Hoping to get some rest while keeping the students busy, Gauss's mathematics teacher asked them to add up the numbers $1 + 2 + \cdots + 100$. The seven-year-old Gauss instantly found out the answer to be 5050. (*Comment.* The historical accuracy of this legendary story is questionable. In [1], Brian Hayes tries to find the origins of this story.)

Figure 1

Keywords: Gauss, triangular numbers

How could Gauss have added a hundred numbers so quickly? One possibility is, he wrote the sum *twice*, once in the forward direction and then in the backward direction:

$$\begin{array}{ccccccccccc} 1 & + & 2 & + & 3 & + & \cdots & + & 99 & + & 100 \\ 100 & + & 99 & + & 98 & + & \cdots & + & 2 & + & 1 \end{array}$$

Observe that $1 + 100 = 101$, $2 + 99 = 101$, $3 + 98 = 101$, and so on. So adding up the numbers in each column gives:

$$101 \quad + \quad 101 \quad + \quad 101 \quad + \quad \cdots \quad + \quad 101 \quad + \quad 101 \quad \text{(100 times)}$$

which shows that the sum of the first hundred numbers is half of 100×101, which is 5050.

This trick can be used to add up as many numbers as we like. For example, $1 + 2 + ... + 1000$ would be $(1000 \times 1001)/2$, which is 500500.

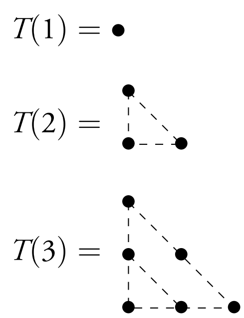
The sum of the first n natural numbers is called the n-th *triangular number*:

$$T(n) = 1 + 2 + ... + n.$$

The first few triangular numbers are:

$$\begin{aligned} T(1) &= 1 \\ T(2) &= 1 + 2 = 3 \\ T(3) &= 1 + 2 + 3 = 6 \\ T(4) &= 1 + 2 + 3 + 4 = 10 \end{aligned}$$

They are called *triangular numbers* because they count the number of objects that can be stacked up in triangles:



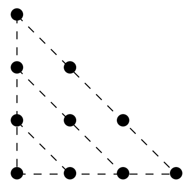
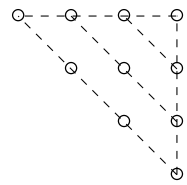
A formula for the n-th triangular number. Using Gauss's trick:

$$\begin{array}{rcccccccccc} T(n) & = & 1 & + & 2 & + & \cdots & + & n-1 & + & n \\ T(n) & = & n & + & n-1 & + & \cdots & + & 2 & + & 1 \\ \hline 2T(n) & = & (n+1) & + & (n+1) & + & \cdots & + & (n+1) & + & (n+1) \end{array}$$

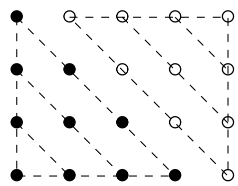
So

$$T(n) = \frac{n(n+1)}{2}. \qquad (1)$$

Explaining the formula using pictures. This formula for $T(n)$ can be explained with pictures as well: For example, the dots in $T(4)$ can be visualized as

$T(4) =$  , or $T(4) =$  ,

so that

$T(4) + T(4) =$  ,

The rectangle on the right has 4 rows and 5 columns. So

$$T(4) + T(4) = 4 \times 5,$$

or $T(4) = (4 \times 5)/2$.

Similarly, for any n, $2T(n)$ points can be arranged to form a rectangular array with n rows and $n + 1$ columns, giving us $2T(n) = n \times (n + 1)$, which we had earlier obtained using Gauss's trick.

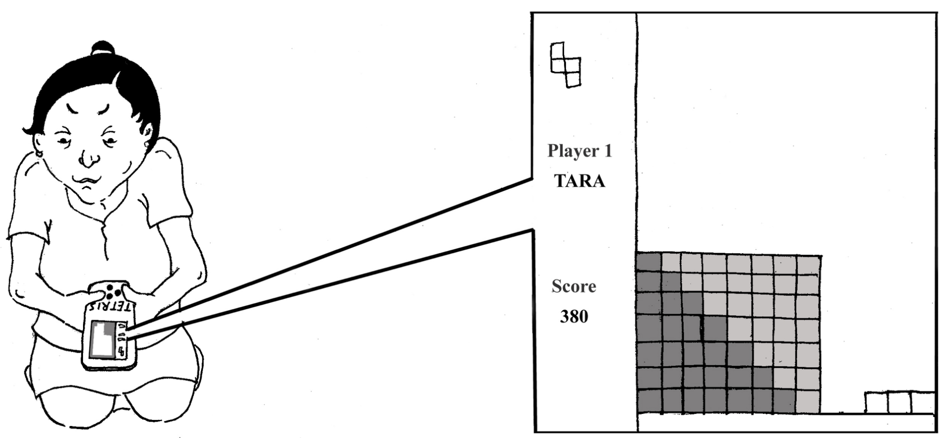


Figure 2

Why would we be interested in triangular numbers? Besides outwitting sadistic teachers, are there other reasons why we should be interested in triangular numbers? To see a couple of examples, try these exercises:

Exercise. In a *round robin tournament* each team plays every other team exactly once. For example, the 32 teams in the FIFA world cup are divided into eight groups of 4 teams each. In the qualifying round, each group plays a round-robin tournament. For example, Group A in the 2014 FIFA world cup had Brazil, Croatia, Mexico and Cameroon. How many matches were played in Group A during the qualifying round? What if the world cup format is changed so that each group has six teams. How many matches will have to be played within each group in the qualifying round?

Solution. Brazil must play each of Croatia, Mexico and Cameroon. This requires three matches. Now let's come to Croatia—we have already taken care of Brazil versus Croatia. It remains for Croatia to play Mexico and Cameroon—that's two more matches. Finally we need a match between Mexico and Cameroon to complete the round robin. Each ✓ in the following table represents a game to be played in this group:

	Brazil	Croatia	Mexico	Cameroon
Brazil		✓	✓	✓
Croatia			✓	✓
Mexico				✓
Cameroon				

Looks familiar, doesn't it? This is the triangular number $T(3)$. Similarly, if there were six teams in a group, the number of games would be $T(5)$.

Exercise. Tara's job is to set up a high-speed network of 24 computers at the university. Each of these computers is required to be connected directly to all the other computers in the network. How many cables will Tara need?

Figure 3

Solution. The first computer will need to be connected to 23 other computers. Once this is done, the second computer will need to be connected to 22 other computers (it is already connected to the first). After this, the third computer will need to be connected to 21 other computers (it is already connected to the first two), and so on. The total number of cables needed will be

$$23 + 22 + \cdots + 2 + 1 = T(23) = \frac{23 \times 24}{2} = 276.$$

How do we know if a number is triangular? Here is a recipé: if $8N+1$ is a perfect square, then N is triangular. Otherwise N is not triangular. For, example, for the first few triangular numbers, we have:

N	1	3	6	10	15	21
$8N+1$	9	25	49	81	121	169

Indeed, if $8N+1$ is the square of a positive integer M, then

$$8N+1 = M^2,$$

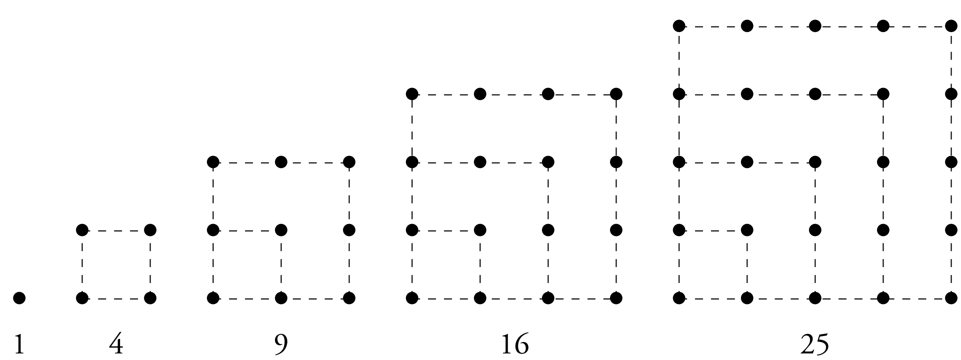
and M has to be odd, because its square is odd. It follows that $M-1$ is even, so $n = (M-1)/2$ is a positive integer. We have

$$\begin{aligned} T_n &= \frac{n(n+1)}{2} \\ &= \frac{M-1}{2} \times \frac{M+1}{2} \times \frac{1}{2} \\ &= \frac{M^2-1}{8} \\ &= N. \end{aligned}$$

This shows that each odd perfect square corresponds to a triangular number, and each triangular number corresponds to an odd perfect square.

Numbers based on other shapes

There are other sequences of integers based on geometrical shapes. The square numbers:



are given by the simple formula

$$S(n) = n^2,$$

while the pentagonal numbers:

$$1, \ 5, \ 12, \ 22, \ 35, \ \ldots$$

come from drawing pentagons of increasing size.

Figure 4

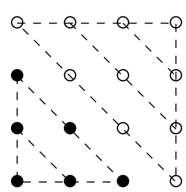
Pentagonal numbers also come with a nice formula:

$$P(n) = \frac{n(3n-1)}{2}.$$

These number sequences sometimes have relationships between them. The sum of two consecutive triangular numbers is always a square number:

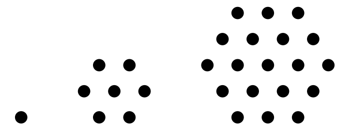
$$T(n) + T(n+1) = (n+1)^2.$$

This can be illustrated geometrically by merging triangles. For example, the figure below illustrates that $T(3) + T(4) = 4^2$.



Exercise. When Gauss dashed his mathematics teacher's hopes for a long break with his quick thinking, the teacher came up with another problem for Gauss to solve. "OK Smart Alec!" he said. "Let's see how you get around this one. Add up the first hundred *odd numbers*." So now they had to add $1 + 3 + 5 + \cdots + 199$. Barely a moment had passed when Tara's eyes lit up and a smile crossed her face. She whispered something to Gauss, who laughed out loud when he realized they'd outwitted the teacher again! Can you figure out what Tara told Gauss?

Exercise. Let's explore two more number sequences that arise from counting the dots in geometric figures. The first is the sequence of *centered hexagonal* numbers:



Can you find a formula for the n-th hexagonal number in terms of the triangular numbers?

Exercise. Have you ever seen a fruit stand selling guavas, *mosambis*, or pomegranates? Can you remember how the vendor stacked the fruits? It's very likely they were stacked in square-based pyramids. How many fruits are needed to make a square pyramid with n levels of fruits? Do you know of a closed formula for this number?

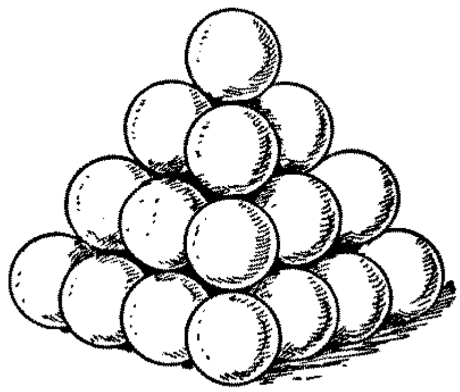


Figure 5

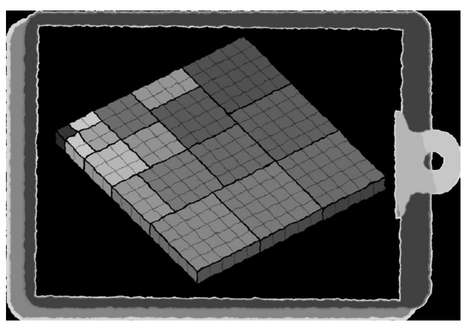
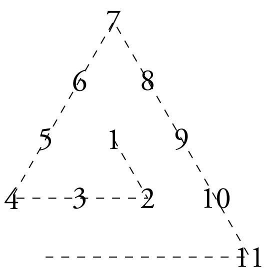


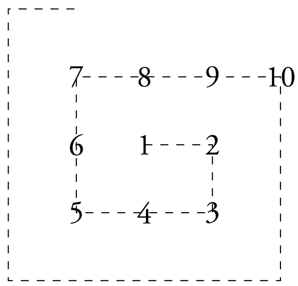
Figure 6

Exercise. Gauss and Tara spend the entire night discussing integer sequences, and by morning they have come up with a startling formula for the sum of the first n cubes: $1^3 + 2^3 + 3^3 + 4^3 + \cdots + n^3$. The sum is just $\left(T(n)\right)^2$, the square of the n-th triangular number! They decide it's their turn to quiz the teacher, so the next day they present their findings and demand a proof from him. Can *you* help the hapless teacher, using the hint from Tara's notes (Figure 6)?

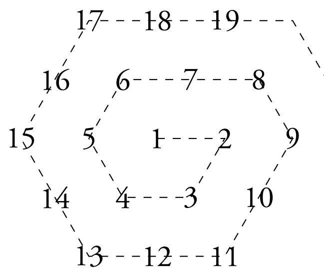
Exercise. After the school year ends, Tara gets a job as an urban planner for soon-to-be-built *Smart Spiral City*. Her boss wants the city plan to have equally spaced buildings, all lying on a single road that spirals outwards. He says that way they'll be identifiable via a single unique address number, with the numbers increasing outwards from the center. Tara considers three possible city plans:



Plan A



Plan B



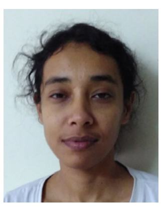
Plan C

Can you help Tara complete the three plans, giving addresses up to the number 50 in each?

Exercise. Tara's boss decides to go with plan A, and gives Tara another task. She must add avenues – roads that are perfectly straight – that will cut through the spiral road, so that the higher numbered houses are easier to reach. These avenues can be oriented in any direction. It turns out Tara's boss and his friends have already purchased houses whose addresses are triangular numbers. How many avenues need to be built in order that all the houses with 'triangular' addresses lie on avenues. How many avenues would have been required in plan B to touch all the square numbers, and in plan C to touch all the hexagonal numbers?

References

1. Brian Hayes, "Sides and Area of Pedal Triangle", *The American Scientist*, Vol. 94, No. 3, May-June 2006, page 200. http://dx.doi.org/10.1511/2006.3.200



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