RAMANUJAN AND SOME ELEMENTARY MATHEMATICAL PROBLEMS

Features

UTPAL MUKHOPADHYAY

The year 1914 is a milestone in the history of Indian mathematics. On 17 March of that memorable year, the legendary Indian mathematician Srinivasa Ramanujan (1887–1920) set off on his historic voyage boarding the ship S S Nevasa for pursuing mathematical research under the supervision of the famous Cambridge mathematician, Professor G. H. Hardy (1877–1947). A detailed account of the historical background of the nourishment of the budding genius Ramanujan has been given by the present author elsewhere (*Resonance*, June 2014). The purpose of the present article is to narrate some interesting episodes of Ramanujan's life and to provide a few elementary problems solved by him to demonstrate how he attacked those problems and using his intuition generalized the results in some cases. Since the two words 'Ramanujan' and 'mathematics' cannot be separated, let us start with an episode involving an interesting mathematical problem.

Srinivasa Ramanujan

Prof G H Hardy

Keywords: Ramanujan, Hardy, Mahalanobis, door number problem, continued fraction, triangular number, square number, FRS, Cambridge, Trinity

In December 1914, when P. C. Mahalanobis (1893-1972) was in England for his studies, he showed Ramanujan, who was also engaged at that time in mathematical research under the supervision of Professor G. H. Hardy, a mathematical problem entitled 'Puzzles at a Village Inn' published in the popular English magazine *Strand*. The problem goes like this:

> *The houses on one side of a street are numbered from 1 to n. The sum of the numbers of the houses lying to the left of a particular house numbered m is equal to the sum of the numbers of the houses lying to the right of that particular house. If n lies between 50 and 500, then calculate the values of m and n.*

It may be mentioned here that being a member of a conservative Brahmin family, Ramanujan cooked his own meals. When Mahalanobis posed the problem to Ramanujan, the latter was engaged in preparing a meal. Without interrupting his work, Ramanujan instantly solved the problem. When Mahalanobis asked Ramanujan how he had arrived at the solution, he answered: "*Immediately I heard the problem it was clear that the solution should be a continued fraction; I then thought, which continued fraction? And the answer came to my mind*". Before presenting the solution of the problem, let us introduce the idea of a 'perfect median'.

The definition of perfect median as given by Brian Hayes in *American Scientist* (Vol. 96, page 36, 2008) is this: *A number m is a perfect median of n consecutive natural numbers* $1, 2, 3, \ldots, m, \ldots n$ *if* $1 + 2 + 3 + \ldots + (m-1) = (m+1) + (m+2) + (m+3) + \ldots + n$, *i.e., if*

$$m^2 = \frac{n(n+1)}{2}. \tag{1}$$

From (1), it is clear that for a perfect median, a square number and a triangular number (see Box 1) must be equal.

Multiplying both sides of (1) by 8 and adding 1 to both sides we get,

$$(2n+1)^2 - 8m^2 = 1 \tag{2}$$

Put:

$$2n + 1 = p, 2m = q. \tag{3}$$

From (2) we get $p^2 - 2q^2 = 1$, or by factorisation,

$$\left(p + \sqrt{2}q\right)\left(p - \sqrt{2}q\right) = 1. \tag{4}$$

Of the infinitely many solutions of (4), the one with the least values of p and q are $p_1 = 3$ and $q_1 = 2$. Using these values in (3), we get: $n_1 = 1$, $m_1 = 1$.

Putting these smallest values of p and q in (4) we obtain,

$$\left(3 + 2\sqrt{2}\right)\left(3 - 2\sqrt{2}\right) = 1. \tag{5}$$

Squaring each bracketed expression of (5) separately, we get,

$$\left(17 + 12\sqrt{2}\right)\left(17 - 12\sqrt{2}\right) = 1.$$

So, the next solution is: $p = 17$, $q = 12$, which yields $n_2 = 8$, $m_2 = 6$.

Again, by cubing (5) we get,

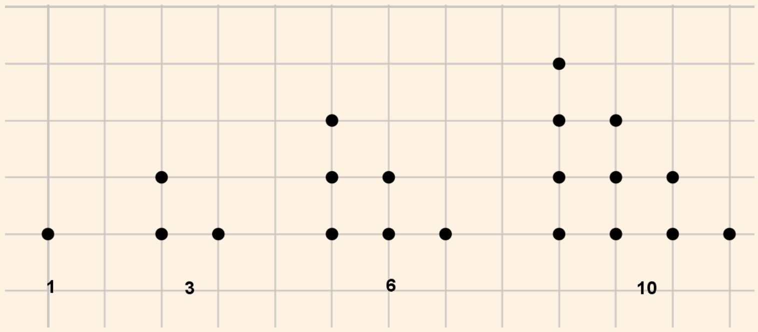
$$\left(99 + 70\sqrt{2}\right)\left(99 - 70\sqrt{2}\right) = 1.$$

So $p_3 = 99$, $q_3 = 70$, and $n_3 = 49$, $m_3 = 35$.

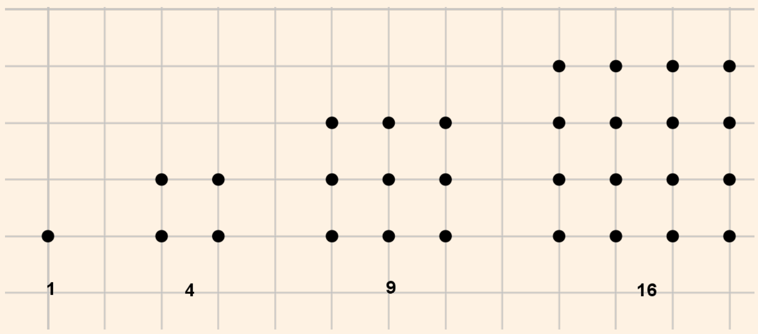
Proceeding in this way, by taking higher powers of (5), we get more solutions of (4).

Triangular and Square Numbers

Numbers which can be geometrically represented by triangular arrays of dots are called triangular numbers. For example, 1, 3, 6, 10, etc., are triangular numbers.



Similarly, numbers whose geometric representations are in the form of a square are square numbers, e.g. 1, 4, 9, 16, etc.



— If the n-th triangular number be T_n then $T_n = n(n+1)/2$.

— If the n-th square number be S_n then $S_n = n^2$.

Box 1

It may be mentioned here that in 1733, Leonhard Euler (1707-1783) in his paper *On the Solutions of Problems of Diophantus About Large Numbers* posed the problem of finding triangular numbers which also happen to be square numbers. His answer was the following: "Triangular numbers with $n = 1, 8, 49, 288, 1681, 9800$, etc., are square numbers corresponding to $m = 1, 6, 35, 204, 1189, 6930$, etc., respectively". For example, if we choose $n = 8$ and $m = 6$, we get the number 36 which is the eighth triangular number and also the sixth square number.

Now let us come back to Ramanujan's solution of the problem. He mentally calculated the first two values 1 and 6 of m and then constructed the following continued fraction (see Box 2 for what 'continued fraction' means):

$$\frac{1}{6-}\frac{1}{6-}\frac{1}{6-}\frac{1}{6-}\frac{1}{6-}\cdots$$

From each convergent (see Box 2) of the above continued fraction, we get two values of m. So the first convergent is $1/6$ and the two values of m are 1 and 6. The second convergent is $6/35$ and the corresponding values of m are 6 and 35. The next convergent is 35/204 and the corresponding values of m are 35 and 204, and so on. If we assume

$$x = \frac{1}{6-}\frac{1}{6-}\frac{1}{6-}\frac{1}{6-}\frac{1}{6-}\dots,$$

then $x = 1/(6-x)$ and therefore $6x - x^2 = 1$, i.e.,

$$x^2 - 6x + 1 = 0. \tag{6}$$

Solving (6), we get $x = 3 \pm 2\sqrt{2}$.

As $x < 1$, the plausible value of x is $3 - 2\sqrt{2}$.

Analyzing the above problem, we get a glimpse into how Ramanujan's mind worked!

Note from the editors: Another way of solving the door number problem is described in the article by Bodhideep Joardar, elsewhere in this issue.

Continued Fractions

An expression of the form shown below is called a **continued fraction**:

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \dots}}}.$$

The same continued fraction can also be written more conveniently as

$$1 + \frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\dots;$$

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ etc. are called the elements of the continued fraction.

The value of the continued fraction obtained by truncating it at a certain stage is called a **convergent**. For the continued fraction shown above, the successive convergents are:

$$1, \frac{3}{2}, \frac{10}{7}, \dots.$$

Box 2

Ramanujan numbers

The Ramanujan number 1729 is well known to us. So, it is not necessary to repeat the story involving Ramanujan and his mentor Hardy and the taxicab. It suffices to say that this is the smallest positive integer which can be expressed as a sum of two cubes in two different ways. This property of the number 1729 was discovered by the French mathematician Bernard Frenicle de Bessy (1604-1674) in 1657. The number is found in one of Ramanujan's notebooks some years before the Hardy-Ramanujan incident. So most probably Ramanujan was already familiar with the special properties of that number. It should be added here that if we consider only cubes of positive integers, then 1729 is indeed the smallest number with that particular property. However, if cubes of negative integers are also taken into consideration, then 91 is the smallest such number, because

$$91 = 6^3 + (-5)^3 = 4^3 + 3^3.$$

Incidentally, 91 is a divisor of 1729. Now one may ask, which numbers larger than 1729 have the same property, i.e., they can be expressed in two different ways as a sum of two cubes? Such numbers are now known as *Ramanujan numbers*. Here are two other numbers with this property:

$$4104 = 2^3 + 16^3 = 9^3 + 15^3;$$
$$20683 = 10^3 + 27^3 = 19^3 + 24^3.$$

Note that if n is a Ramanujan number, then (trivially) so is nk^3 for any positive integer k; so $1729k^3$, $4104k^3$ and $20683k^3$ are Ramanujan numbers for any positive integer k.

A general rule for generating Ramanujan numbers is given below. Let m, n be integers. Define the integers a, b, c, d by using the following formulas:

$$a = 1 - (m - 3n)\left(m^2 + 3n^2\right),$$
$$b = -1 + (m + 3n)\left(m^2 + 3n^2\right),$$
$$c = (m + 3n) - \left(m^2 + 3n^2\right)^2,$$
$$d = -(m - 3n) + \left(m^2 + 3n^2\right)^2.$$

Then the following is identically true:

$$a^3 + b^3 = c^3 + d^3.$$

By appropriate rearrangement of terms (in case any of a, b, c, d turn out to be negative numbers), we get a Ramanujan-type relation involving only positive integers. Of course, we suitably cancel out on common factors. Some examples are shown below.

Example 1
Let $m = 2, n = 1$; then we get $a = 8, b = 34, c = -44, d = 50$. This yields the relation $8^3 + 34^3 = (-44)^3 + 50^3$. This in turn yields the interesting relation

$$4^3 + 17^3 + 22^3 = 25^3.$$

Example 2
Let $m = 3, n = 1$; then we get $a = 1, b = 71, c = -138, d = 144$. This yields the relation

$$1^3 + 71^3 + 138^3 = 144^3.$$

Example 3
Let $m = 4, n = 1$; then we get $a = -18, b = 132, c = -354, d = 360$. On division by 6, we get the following relation: $(-3)^3 + 22^3 = (-59)^3 + 60^3$, which may be rewritten as

$$59^3 + 22^3 = 60^3 + 3^3.$$

This yields the Ramanujan number 216027.

Example 4
Let $m = 1, n = 1$; then $a = 3, b = 5, c = -4, d = 6$ and hence the following relation:

$$3^3 + 4^3 + 5^3 = 6^3.$$

So if three solid metallic spheres of radii 3 cm, 4 cm and 5 cm are melted to form a larger sphere, then the radius of the new sphere will be 6 cm.

A Geometric Corollary of Ramanujan

Now, let us cite a particular work of Ramanujan which may be useful for school students. Ramanujan deduced the following result as a corollary of Pythagoras theorem.

Two segments BQ and CP are cut off from the hypotenuse BC of the right angled triangle ABC such that $BQ = BA$ *and* $CP = CA$. *Then* $PQ^2 = 2BP \cdot QC$.

Proof: We have (see Figure 1):

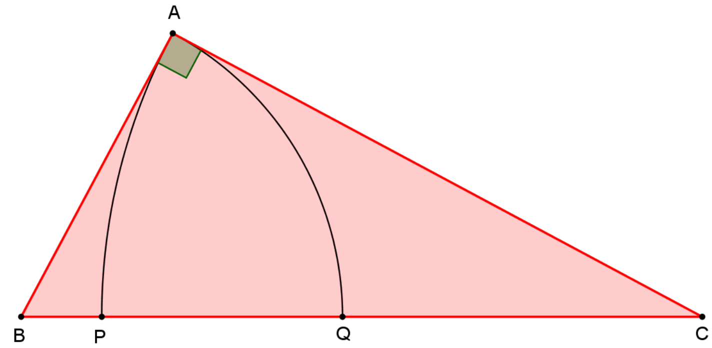


Figure 1

$BA = BQ = BP + PQ,$

so

$$AB^2 = BP^2 + PQ^2 + 2BP \cdot PQ. \tag{7}$$

Similarly,

$$AC^2 = CQ^2 + PQ^2 + 2CQ \cdot PQ. \tag{8}$$

Since BC is the hypotenuse of the right-angled triangle ABC, $BC^2 = AB^2 + AC^2$, so:

$$\begin{aligned} BC^2 &= BP^2 + PQ^2 + 2BP \cdot PQ + CQ^2 + PQ^2 + 2CQ \cdot PQ \\ &= BP^2 + 2PQ^2 + CQ^2 + 2BP \cdot PQ + 2CQ \cdot PQ. \end{aligned} \tag{9}$$

Again, $BC^2 = (BP + PQ + QC)^2$, so:

$$BC^2 = BP^2 + PQ^2 + CQ^2 + 2BP \cdot PQ + 2CQ \cdot PQ + 2BP \cdot CQ. \tag{10}$$

From (9) and (10) it follows that $PQ^2 = 2BP \cdot QC$, as required.

An algebraic identity

Using the above corollary, Ramanujan arrived at an algebraic identity. Let us put $AB = a$ and $AC = b$. Then $BC = (a^2 + b^2)^{1/2}$. Next, $BQ = AB = a$ and $CP = AC = b$, so

$$\begin{aligned} PQ &= BQ + CP - BC = a + b - \left(a^2 + b^2\right)^{1/2}, \\ QC &= BC - BQ = \left(a^2 + b^2\right)^{1/2} - a. \end{aligned}$$

A similar expression can be given for BP. Since $PQ^2 = 2BP \cdot QC$, one obtains,

$$\left(a + b - \left(a^2 + b^2\right)^{1/2}\right)^2 = 2\left(\left(a^2 + b^2\right)^{1/2} - a\right) \cdot \left(\left(a^2 + b^2\right)^{1/2} - b\right). \tag{11}$$

Extending the result (11), Ramanujan arrived at the following identity of the third degree:

$$\left((a + b)^{2/3} - \left(a^2 - ab + b^2\right)^{1/3}\right)^3 = 3\left(\left(a^3 + b^3\right)^{1/3} - a\right) \cdot \left(\left(a^3 + b^3\right)^{1/3} - b\right). \tag{12}$$

It can be shown that both sides of (12) are equal to

$$3\left(a^3+b^3\right)^{2/3}-(a+b)\left(a^3+b^3\right)^{1/3}+ab.$$

The above results are nice examples of the interrelationship between two branches of mathematics (in this case, geometry and algebra), as demonstrated by the great genius Ramanujan. Such examples may be cited during classroom teaching to inspire students in their search for deeper mathematical results.

Two problems posed and solved by Ramanujan

Next we reproduce below two problems and their solutions published in *Indian Journal of Mathematics* and their solutions by Ramanujan to show his ingenuity in elementary mathematics as well.

Problem 1

Evaluate $$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\ldots}}}}}.$$

Ramanujan's Solution

We know that $n+2=\sqrt{1+(n+1)(n+3)}$; hence:

$$n(n+2)=n\sqrt{1+(n+1)(n+3)}.$$

Let $f(n)=n(n+2)$. The above relation may then be written as:

$$f(n)=n\sqrt{1+f(n+1)}.$$

This substitution may be repeated iteratively:

$$\begin{aligned} f(n)&=n\sqrt{1+f(n+1)}\\ &=n\sqrt{1+(n+1)\sqrt{1+f(n+2)}}=\ldots\\ &=n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\sqrt{1+\ldots}}}}. \end{aligned}$$

Putting $n=1$ in the above relation, we get:

$$3=\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\ldots}}}}}.$$

Problem 2

Evaluate:

$$\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\ldots}}}}.$$

Ramanujan's Solution

We know that $n(n+3)=n\sqrt{(n+5)+(n+1)(n+4)}$.

Let $f(n)=n(n+3)$; then

$$f(n)=n\sqrt{(n+5)+f(n+1)}.$$

We now use this identity iteratively:

$$\begin{aligned} f(n) &= n\sqrt{(n+5)+f(n+1)} \\ &= n\sqrt{(n+5)+(n+1)\sqrt{(n+6)+f(n+2)}} = \ldots \\ &= n\sqrt{(n+5)+(n+1)\sqrt{(n+6)+(n+2)\sqrt{(n+7)+(n+3)\sqrt{(n+8)+\ldots}}}}. \end{aligned}$$

Putting $n=1$ in the above relation, we get:

$$4 = \sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\ldots}}}}.$$

These two solutions demonstrate the deep insight and high level of intuition possessed by Ramanujan. We should mention here that in the book '*Wonders of Numbers*' by Clifford A. Pickover, the following formula is described as the most beautiful formula discovered by Ramanujan:

$$1+\frac{1}{1.3}+\frac{1}{1.3.5}+\frac{1}{1.3.5.7}+\ldots+1+\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\ldots=\sqrt{\frac{\pi e}{2}}.$$

Ramanujan becoming an FRS

Let us end our discussion by mentioning an episode related to Ramanujan and some information about his research work.

Prof. Hardy not only shaped Ramanujan's research career, but also exercised his powers to bring recognition for Ramanujan. He believed that his selection as a Fellow of the Royal Society (FRS) was a necessity for Ramanujan to boost his spirit. In Ramanujan's FRS application form, Hardy proposed his name, and the proposal was seconded by Major P. A. McMahon (1854–1929). There were eleven other signatories.[1] Among them, all except McMahon were Wranglers of the Cambridge Mathematical Tripos (with six Senior Wranglers). It is interesting to note that before writing his historic letter (dated 16 January, 1913) to Prof. Hardy, Ramanujan had sent letters to Baker and Hobson mentioned above without receiving any response from them. Anyway, Ramanujan's application form was submitted to the Royal Society on 18 December, 1917. His deteriorating health condition was a source of anxiety for Hardy. So he wasted no time in trying to convince high ranking persons of the Royal Society regarding Ramanujan's mathematical talent. He communicated detailed information to the then President of the Royal Society and Nobel Prize winning physicist Sir J. J. Thompson (1856–1940). Hardy also mentioned in his letter the physical condition of Ramanujan and warned that if the Royal Society delayed Ramanujan's selection, then "*The Society would have to live forever with its failure to honour him.*" All these efforts of Hardy culminated with the selection of Ramanujan as FRS in the meeting of Royal Society on 28 February, 1918. After 2 May, 1918 Ramanujan was entitled to write FRS after his name, becoming the second Indian to achieve this honour. It may be mentioned here that prior to his selection as FRS, Ramanujan was selected Member of London Mathematical Society on 6 December, 1917 and Member of Cambridge Philosophical Society on 18 February, 1918. On 10 October, 1918, Ramanujan was selected as a Fellow of Trinity College, being the first Indian to achieve this honour. In Ramanujan's selection as a Fellow of Trinity College, Littlewood played a leading role for nullifying the racial issues raised against Ramanujan.

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[1] J. H. Grace (1873-1958), Joseph Larmour (1857-1942), T. J. Bromwich (1875-1929), E. W. Hobson (1856-1933), H. F. Baker (1866-1956), J. E. Littlewood (1885-1977), J. W. Nicholson (1881-1955), W. H. Young (1863-1942), E. T. Whittaker (1873-1956), A. R. Forsyth (1858-1942) and A. N. Whitehead (1861-947)

When questions arose whether Ramanujan was mentally fit, Littlewood produced two medical certificates to prove Ramanujan's mental fitness. The main argument placed in favour of Ramanujan was "*For a Fellow of Royal Society to be denied a Trinity Fellowship would be a scandal*." All these honours encouraged Ramanujan in his mathematical research. According to Srinivas Rao, "*These awards acted as great incentives to Ramanujan who discovered some of the most beautiful results in mathematics subsequently*." Justification of this statement can be judged from the fact that shortly before his selection to the Royal Society, Ramanujan jumped in front of a running train to kill himself but was saved somehow by the alertness of the train driver. So, no doubt Ramanujan was suffering from some kind of depression before achieving the honours mentioned above. He was arrested by Scotland Yard Police but was released through Hardy's intervention.

Some Statistical Information

We may get an idea of Ramanujan's mathematical work during his student life and immediately after his failure in F. A. Examination. In 1902, Ramanujan mastered the method of solving cubic equations and used it to develop his own method of solving quadratic equations. In 1903, he failed to solve equations of fifth degree due to his ignorance regarding the impossibility of its solution in simple form. During 1904, Ramanujan paid attention to the summation of series of the form $\sum \frac{1}{n}$ and calculated the value of the Euler constant to 15 decimal places. In the year 1908, he engaged himself in mathematical works on continued fractions and convergent series. Before going abroad, five research papers of Ramanujan were published in the *Journal of The Indian Mathematical Society*. The title of Ramanujan's first published paper was *Some properties of Bernoulli Numbers* (vol. 3, pp 219 – 234, 1911). The other four papers were *On question 330 of Prof. Sanjana* (vol. 4, pp 59 – 61, 1912), *Notes on a Set of Simultaneous Equations* (vol. 4, pp 94 – 96, 1913), *Irregular Numbers* (vol. 5, pp 105 – 106, 1913) and *Squaring the Circle* (vol. 5, page 132, 1913). Afterwards, the number of papers of Ramanujan published in the years 1914, 1915, 1916, 1917, 1918, 1919, 1920 and 1921 were 1, 9, 3, 7, 4, 4, 3 and 1 respectively. Ramanujan's last paper *Congruence Properties of Partitions* was published posthumously in *Mathematische Zeitschrift* (vol.93, pp 147 – 150, 1921). In all, 37 papers of Ramanujan were published in his research career, seven of them being joint papers with Prof. Hardy. Among those 37 papers, 11 were published in the *Journal of The Indian Mathematical Society*, 7 in *Proceedings of the London Mathematical Society*, 6 in *Messenger of Mathematics*, 5 in *Proceedings of the Cambridge Philosophical Society*, 3 in *Quarterly Journal of Mathematics*, 2 in *Transactions of the Cambridge Philosophical Society*, and 1 each in *Proceedings of the Royal Society*, *Mathematische Zeitschrift* and *Comptes Rendus*.

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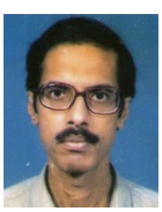

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