Stirling SET NUMBERS & Powers of INTEGERS

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Introduction

This article introduces a remarkable class of combinatorial numbers, the ***Stirling set numbers***. They are also known as *Stirling numbers of the second kind*. (In the literature, you will find many references to this term. Note that there are also *Stirling numbers of the first kind*.) However, we shall not use this name; the name 'Stirling set number' seems more natural. These numbers are named after the 18th century Scottish mathematician James Stirling, and they are the natural counterparts of the binomial coefficients. Some readers may recall the name 'Stirling'. Indeed, it is the same Stirling whose name features in "Stirling's approximation for the factorial function."

Stirling set numbers feature prominently in the problem of finding a formula (in terms of n) for the sum of the k-th powers of the first n natural numbers, for given positive integer values of k, which is why we have included this article in this issue. (See the articles by Prof Tikekar and by Prof Joshi.)

Defining the Stirling set numbers

We start by defining the numbers. For positive integers k and r, the ***Stirling set number*** $S(k, r)$, is the *number of ways of partitioning the set* $\{1, 2, 3, \ldots, k\}$ *into* r *non-empty subsets* (order does not matter). Equivalently, $S(k, r)$ is the number of ways to put k distinct objects into r non-distinct boxes in such a way that no box is empty. Thus, for example:

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- $S(3,2)$ is the number of ways to partition a 3-element set into two nonempty subsets. It is easy to see that $S(3,2) = 3$, by simply listing all the possibilities. For, if the set is $\{a, b, c\}$, then it can be partitioned into two nonempty subsets in the following ways: $\{a, b\} \cup \{c\}$; $\{a, c\} \cup \{b\}$; and $\{b, c\} \cup \{a\}$.
- $S(4,2)$ is the number of ways to partition a 4-element set into two nonempty subsets. Let us compute by hand the value of $S(4,2)$. Let the set be $\{a, b, c, d\}$. Since 4 can be written as a sum of two positive integers as $2+2$ and $3+1$, there are two kinds of partitions: (i) the two subsets have 2 elements each; (ii) one subset has 3 elements and the other subset has 1 element. It is not difficult to see that the first option contains three possibilities (namely: $\{a, b\} \cup \{c, d\}$; $\{a, c\} \cup \{b, d\}$; $\{a, d\} \cup \{b, c\}$), while the second option contains four possibilities (namely: $\{a, b, c\} \cup \{d\}$; $\{a, b, d\} \cup \{c\}$; $\{a, c, d\} \cup \{b\}$; $\{b, c, d\} \cup \{a\}$). Hence $S(4,2) = 7$.

More such values can be found. More generally, we have, for any positive integer k,

$$\left.\begin{aligned} S(k,1) &= 1, \\ S(k,k) &= 1, \\ S(k,k-1) &= \binom{k}{2}. \end{aligned}\right\} \tag{1}$$

The first two equalities are obvious.

To see why the third is true, observe that if $r = k-1$, then one of the $k-1$ subsets has two elements, while the other subsets have one element each. Since the subsets are indistinguishable, this can be done in as many ways as the number of ways of selecting two elements from the given set. Hence the stated equality.

Next:

$$S(k,2) = 2^{k-1} - 1. \tag{2}$$

For: $r = 2$ means that we partition the given set of k objects into two non-empty subsets; the order does not matter. Arbitrarily put aside any one object (it does not matter which one); call it X. Each of the remaining $k-1$ objects can choose to partner with X or not; this yields 2^{k-1} choices. However, we cannot have all of the objects partnering with X, as that would result in only one subset and not two, as required. So we need to subtract 1 from the number obtained. Hence the number of ways is $2^{k-1} - 1$. That is, $S(k,2) = 2^{k-1} - 1$. (Observe that this is consistent with our earlier finding that $S(4,2) = 7$.)

Values of $S(k,r)$ for $2 < r < k-1$ may be found using the following very convenient recurrence relation:

$$S(k,r) = r\,S(k-1,r) + S(k-1,r-1). \tag{3}$$

To see why (3) is true, consider where object k belongs. There are two possibilities:

- If k is in a subset all by itself, with no partners, then the remaining subsets correspond to a way of partitioning the set $\{1, 2, \ldots, k-1\}$ into $r-1$ nonempty subsets. This can be done in $S(k-1, r-1)$ ways.
- If on the other hand, k lies in a subset with more than 1 element, then by hiding just that element, we get a partition of the set $\{1, 2, \ldots, k-1\}$ into r nonempty subsets. This can be done in $S(k-1, r)$ ways. Now bring back k. It can be put into any of the r subsets; so we get $rS(k-1,r)$ ways.

Hence $S(k,r) = r\,S(k-1,r) + S(k-1,r-1)$, as claimed.

Values of the Stirling set numbers have been displayed in Figure 1. They have been computed using the recursive formula (3).

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$
$k=1$	1							
$k=2$	1	1						
$k=3$	1	3	1					
$k=4$	1	7	6	1				
$k=5$	1	15	25	10	1			
$k=6$	1	31	90	65	15	1		
$k=7$	1	63	301	350	140	21	1	
$k=8$	1	127	966	1701	1050	266	28	1

Figure 1. The Stirling set numbers $S(k, r)$

Observe that if we put $r = 2$ in this recurrence relation, then we get

$$S(k, 2) = 2\,S(k-1, 2) + S(k-1, 1) = 2\,S(k-1, 2) + 1,$$

and this allows us to prove inductively that $S(k, 2) = 2^{k-1} - 1$, thereby yielding another proof of that relation.

Connection between Stirling set numbers and powers of integers

There is an important relation that allows us to express the quantity n^k in terms of the Stirling set numbers:

Theorem 1. *For positive integers n and k, the following identity is true:*

$$n^k = \sum_{r=1}^{k} r! \cdot S(k, r) \cdot \binom{n}{r}. \tag{4}$$

This relation needs to be understood. It helps if we consider small values of k.

k = 1: The sole Stirling set number for $k = 1$ is $S(1, 1) = 1$, so the claim reduces to the trivial statement $n = 1 \cdot n$.

k = 2: The Stirling set numbers for $k = 2$ are $S(2, 1) = 1$, $S(2, 2) = 1$. Hence the claim reduces to the following statement:

$$n^2 = 1 \cdot 1 \cdot \binom{n}{1} + 2 \cdot 1 \cdot \binom{n}{2},$$

i.e.,

$$n^2 = \binom{n}{1} + 2 \cdot \binom{n}{2}, \tag{5}$$

which may be verified algebraically or using combinatorial reasoning. (Here is a combinatorial proof. Consider all ordered pairs (x, y), where both x and y are elements of $\{1, 2, 3, \ldots, n\}$. The total number of such pairs is n^2. Now subdivide the pairs (x, y) into two categories: those in which $x = y$, and those in which $x \neq y$. The number of pairs of the first kind is $\binom{n}{1} = n$, and the number of pairs of the second kind is $2 \cdot \binom{n}{2}$. Hence the stated result.)

k = 3: The Stirling set numbers for $k = 3$ are $S(3,1) = 1$, $S(3,2) = 3$, $S(3,3) = 1$. Hence the claim reduces to the following statement:

$$n^3 = 1 \cdot 1 \cdot \binom{n}{1} + 2 \cdot 3 \cdot \binom{n}{2} + 6 \cdot 1 \cdot \binom{n}{3},$$

i.e.,

$$n^3 = \binom{n}{1} + 6 \cdot \binom{n}{2} + 6 \cdot \binom{n}{3}, \tag{6}$$

which too may be verified algebraically or using combinatorial reasoning. (Here is a combinatorial proof, just like the one presented above. Consider all triples (x, y, z), where $x, y, z \in \{1, 2, 3, \ldots, n\}$. The total number of such triples is n^3. Now subdivide the triples (x, y, z) into three categories according to the number of different numbers used in the triple; this could be 1, 2, or 3. The number of triples of the first kind is $\binom{n}{1}$, the number of triples of the second kind is $\binom{n}{2} \times \binom{2}{1} \times \frac{3!}{2!} = 6 \cdot \binom{n}{2}$, and the number of triples of the third kind is $6 \cdot \binom{n}{3}$. Hence the stated result.) (*Remark.* The second of these three claims may need some justification but we leave the details to the reader.)

k = 4: The Stirling set numbers for $k = 4$ are $S(4,1) = 1$, $S(4,2) = 7$, $S(4,3) = 6$, $S(4,4) = 1$. Hence the claim reduces to the following statement:

$$n^4 = 1 \cdot 1 \cdot \binom{n}{1} + 2 \cdot 7 \cdot \binom{n}{2} + 6 \cdot 6 \cdot \binom{n}{3} + 24 \cdot 1 \cdot \binom{n}{4},$$

i.e.,

$$n^4 = \binom{n}{1} + 14 \cdot \binom{n}{2} + 36 \cdot \binom{n}{3} + 24 \cdot \binom{n}{4}. \tag{7}$$

Again, this may be verified algebraically or using combinatorial reasoning. (Here is a combinatorial proof. Consider all four-tuples (x, y, z, w), where $x, y, z, w \in \{1, 2, 3, \ldots, n\}$. The total number of such four-tuples is n^4. Now subdivide these four-tuples into four categories according to the number of different numbers used in the four-tuple; this could be 1, 2, 3, or 4. The number of four-tuples of the first kind is $\binom{n}{1}$, the number of four-tuples of the second kind is $\binom{n}{2} \times \left(\frac{4!}{2!2!} + 2 \times \frac{4!}{3!}\right) = 14 \cdot \binom{n}{2}$, the number of four-tuples of the third kind is $\binom{n}{3} \times \binom{3}{1} \times \frac{4!}{2!} = 36 \cdot \binom{n}{3}$, and the number of four-tuples of the fourth kind is $24 \cdot \binom{n}{4}$. Hence the stated result.)

We prove the general result (for the higher powers) later in the article.

There is another set of important relations that we make use of:

Theorem 2. *The following relations are true for any positive integer n:*

$$\binom{n+1}{2} = \binom{n}{1} + \binom{n-1}{1} + \binom{n-2}{1} + \cdots + \binom{1}{1},$$

$$\binom{n+1}{3} = \binom{n}{2} + \binom{n-1}{2} + \binom{n-2}{2} + \cdots + \binom{1}{2},$$

$$\binom{n+1}{4} = \binom{n}{3} + \binom{n-1}{3} + \binom{n-2}{3} + \cdots + \binom{1}{3}, \quad \ldots.$$

The number of terms on the right is n in each case. However, recall that $\binom{r}{s} = 0$ when $s > r$. (So, in effect, the summations stop earlier.)

The relations in Theorem 2 are easily proved using combinatorial reasoning. We do not give the proofs here. Please try to find them for yourself.

Formulas for the sums of powers

Bringing together the results of Theorem 1 and Theorem 2, we obtain formulas for the sums of the squares, the cubes, and the fourth and higher powers of the first n natural numbers. First, the squares. We know that

$$n^2 = \binom{n}{1} + 2 \cdot \binom{n}{2}.$$

Hence, using the identities from Theorem 2:

$$\begin{aligned} \sum_{i=1}^{n} i^2 &= \sum_{i=1}^{n} \binom{i}{1} + 2 \cdot \sum_{i=1}^{n} \binom{i}{2} \\ &= \binom{n+1}{2} + 2 \cdot \binom{n+1}{3}. \end{aligned} \tag{8}$$

On simplifying the expressions on the right side, we get the familiar formula for the sum of the squares of the first n natural numbers.

Similarly, for the sum of the cubes: we know that

$$n^3 = \binom{n}{1} + 6 \cdot \binom{n}{2} + 6 \cdot \binom{n}{3}.$$

Hence, using Theorem 2 as earlier:

$$\begin{aligned} \sum_{i=1}^{n} i^3 &= \sum_{i=1}^{n} \binom{i}{1} + 6 \cdot \sum_{i=1}^{n} \binom{i}{2} + 6 \cdot \sum_{i=1}^{n} \binom{i}{3} \\ &= \binom{n+1}{2} + 6 \cdot \binom{n+1}{3} + 6 \cdot \binom{n+1}{4}. \end{aligned} \tag{9}$$

As earlier, on simplifying the expressions on the right side, we get the familiar formula for the sum of the cubes of the first n natural numbers.

Similarly we get, for the sum of the fourth powers:

$$\sum_{i=1}^{n} i^4 = \binom{n+1}{2} + 14 \cdot \binom{n+1}{3} + 36 \cdot \binom{n+1}{4} + 24 \cdot \binom{n+1}{5}. \tag{10}$$

We can continue in this vein indefinitely.

Proof of Theorem 1

The derivations presented above rest critically on Theorem 1, which we proved only for the cases $k = 2, 3$, and 4. So it is incumbent upon us, now, to provide the full proof of the theorem. Let us recall the statement here.

Claim (Theorem 1). *For positive integers n and k, the following identity is true:*

$$n^k = \sum_{r=1}^{k} r! \cdot S(k, r) \cdot \binom{n}{r}.$$

Proof. Consider all k-tuples whose elements are drawn from the set $\{1, 2, 3, \ldots, n\}$. The number of such k-tuples is n^k. We subdivide these k-tuples according to the number r of different numbers used in the k-tuple, $r \in \{1, 2, \ldots, k\}$.

To motivate the proof, we take a close look at the cases $r = 1, 2, 3$.

r = 1: This means that a single number has been used in the k-tuple, repeated k times. The number of such k-tuples is therefore the same as the number of ways of choosing that number and is thus equal to $\binom{n}{1}$. Note that we may write this in the form $1! \times S(k, 1) \times \binom{n}{1}$

r = 2: This means that just two numbers (call them a and b) have been used repeatedly in the k-tuple. These two numbers can be chosen in $\binom{n}{2}$ ways. In the k-tuple thus obtained, every number is either a or b. The placement of these a's and b's creates a natural partition of the set $\{1, 2, \ldots, k\}$ into two nonempty subsets. The number of such partitions is $S(k, 2)$. For each such partition, we have a choice of deciding which part will be occupied by the a's and which part by the b's; there are 2! ways of making this choice.

Hence the total number of ways corresponding to $r = 2$ is

$$2! \times S(k, 2) \times \binom{n}{2}.$$

r = 3: This means that just three numbers (call them a, b, c) have been used repeatedly in the k-tuple. These numbers can be chosen in $\binom{n}{3}$ ways. In the k-tuple thus obtained, every number is either a or b or c. Their placement creates a natural partition of the set $\{1, 2, \ldots, k\}$ into three nonempty subsets. The number of such partitions is $S(k, 3)$. For each such partition, we have a choice of deciding which part will be occupied by the a's, which part by the b's, and which part by the c's; there are 3! ways of making this choice.

Hence the total number of ways corresponding to $r = 3$ is

$$3! \times S(k, 3) \times \binom{n}{3}.$$

The reasoning described for the cases $r = 2$ and $r = 3$ is perfectly general; clearly, we do not need to repeat it for other values of r. Theorem 1 thus follows. $\square$

As a consequence of the above and the reasoning used earlier, we see that the truth of the following theorem has been established:

Theorem 3. *For positive integers n and k, the following identity is true:*

$$1^k + 2^k + \cdots + n^k = \sum_{r=1}^{k} r! \cdot S(k, r) \cdot \binom{n+1}{r+1}. \tag{11}$$

Concluding remarks

The Stirling set numbers provide a well-knit and intellectually very satisfying way of arriving at formulas for the sums of the k-th powers of the first n natural numbers, for any positive integer k. What is pleasing about this approach is its combinatorial nature and the way it 'hangs together.'

For further study, the reader could refer to the readings suggested. Reference [2] is a particularly good read.

References

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