THE POWER TRIANGLE

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In the companion article by Prof V G Tikekar, a method is described for finding a formula for the sum of the k-th powers of the first n positive integers, for any positive integer k, using a triangular array of numbers—the ***Power Triangle***. This article shows why the method works. The proof makes use of a family of combinatorial numbers called the ***Stirling set numbers*** (also called *Stirling numbers of the second kind*; they are dwelt upon in the other companion article by Dr Shailesh Shirali). By exploiting a relation between these numbers and the power functions, we are led to the formulas we seek.

Recalling the definition. For positive integers k and r, the ***Stirling set number*** $S(k, r)$, is the *number of ways of partitioning the set* $\{1, 2, 3, \ldots, k\}$ *into* r *non-empty subsets*; order does not matter. Note that the definition implies that $S(k, r) = 0$ for $k < r$. For example, $S(3, 2) = 3$ and $S(4, 2) = 7$, as the reader can verify. The following is true about the Stirling set numbers:

- For all positive integers k, we have
$$S(k, 1) = 1, \quad S(k, k) = 1, \quad S(k, 2) = 2^{k-1} - 1.$$
- For all positive integers $k > 1$, we have
$$S(k, k-1) = \binom{k}{2}.$$
- For all positive integers k and r, $1 < r < k - 1$, we have:
$$S(k, r) = r\,S(k-1, r) + S(k-1, r-1). \qquad (1)$$

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	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$
$k=1$	1							
$k=2$	1	1						
$k=3$	1	3	1					
$k=4$	1	7	6	1				
$k=5$	1	15	25	10	1			
$k=6$	1	31	90	65	15	1		
$k=7$	1	63	301	350	140	21	1	
$k=8$	1	127	966	1701	1050	266	28	1

Figure 1. The Stirling set numbers $S(k, r)$

A few values of the Stirling set numbers are displayed in Figure 1. They have been computed using the above recursive relation.

Connection between Stirling set numbers and powers of integers. There is an important relation connecting the quantity n^k with the Stirling set numbers:

Theorem 1. *For positive integers n and k,*

$$n^k = \sum_{r=1}^{k} r! \cdot S(k, r) \cdot \binom{n}{r}. \tag{2}$$

This leads naturally to Theorem 2 which gives a formula for the sum of the k-th powers of the first n natural numbers, for any positive integer k. Both Theorem 1 and Theorem 2 are proved in Shirali's article.

Theorem 2. *For positive integers n and k,*

$$1^k + 2^k + \cdots + (n-1)^k + n^k = \sum_{r=1}^{k} r! \cdot S(k, r) \cdot \binom{n+1}{r+1}. \tag{3}$$

Connection with the Power Triangle. We now recall the method presented in Prof Tikekar's article, involving the power triangle. The rules governing its formation are:

(i) Row n has $n+1$ numbers, $T(n,1)$, $T(n,2)$, $T(n,3)$, …, $T(n, n+1)$; we adopt the convention that $T(n,r) = 0$ if $r < 1$ or if $r > n+1$.

(ii) $T(n, 1) = 1$ for $n = 0, 1, 2, \ldots$.

(iii) For $n = 1, 2, 3 \ldots$ and $r = 1, 2, 3, \ldots, n+1$,

$$T(n,r) = (r-1) \cdot T(n-1, r-1) + r \cdot T(n-1, r). \tag{4}$$

Figure 2 shows the first few rows of the Power Triangle, computed using relation (4).

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$n=0$	1					
$n=1$	1	1				
$n=2$	1	3	2			
$n=3$	1	7	12	6		
$n=4$	1	15	50	60	24	
$n=5$	1	31	180	390	360	120

Figure 2. The first few rows of the Power Triangle

Formula for the sum of the powers. For each positive integer k, define the following function f_k on the set of positive integers $\mathbb{N}$:

$$f_k(n) = 1^k + 2^k + \cdots + (n-1)^k + n^k.$$

A formula for $f_k(n)$ is claimed to be:

$$f_k(n) = \sum_{r=1}^{k+1} \binom{n}{r} \cdot T(k,r). \tag{5}$$

This is what we must now set out to prove.

A formula connecting the Stirling set numbers and the Power Triangle numbers. To make progress, we must find a connection between the Stirling set numbers and the Power triangle numbers. The connection is easy to spot when one places the two sets of numbers next to each other, as in Figure 3:

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$k=1$	1					
$k=2$	1	1				
$k=3$	1	3	1			
$k=4$	1	7	6	1		
$k=5$	1	15	25	10	1	
$k=6$	1	31	90	65	15	1

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$n=0$	1					
$n=1$	1	1				
$n=2$	1	3	2			
$n=3$	1	7	12	6		
$n=4$	1	15	50	60	24	
$n=5$	1	31	180	390	360	120

Figure 3. Stirling set numbers (left) and Power triangle numbers (right)

The formula is easily seen to be this: for all integers $k \geq 1$, $0 \leq r \leq k$,

$$T(k,r) = (r-1)! \cdot S(k+1,r). \tag{6}$$

Once seen, it is easy to prove the relationship using the principle of induction. Thus, assuming that the relation $T(i, r) = (r-1)!\ S(i+1, r)$ is true for $i = 0, 1, \ldots, k-1$ and $r = 1, 2, \ldots, i+1$, we have, by repeatedly using the recursive properties of the Stirling set numbers and the T-numbers,

$$\begin{aligned} T(k, r) &= (r-1) \cdot T(k-1, r-1) + r \cdot T(k-1, r) \\ &= (r-1) \cdot (r-2)! \cdot S(k, r-1) + r \cdot (r-1)! \cdot S(k, r) \\ &= (r-1)! \cdot (S(k, r-1) + r \cdot S(k, r)) \\ &= (r-1)! \cdot S(k+1, r). \end{aligned}$$

This proves the property that had been claimed.

Proof of the Power Triangle formula. With all the pieces now in place, the proof of the claimed relation (5) is now not too difficult to work out. It has been displayed as a separate boxed item (Box 1).

Remarks from the editor.

- It is quite often the case, in combinatorial settings, that once a property has been seen, it is relatively easy to prove it using the principle of induction. Finding the property poses much more of a challenge than proving it! The case of (6) illustrates this remark perfectly.
- Proving (5) using (6) and the recursive properties of the Stirling set numbers and the T-numbers involves a considerable amount of algebraic manipulation; in particular, manipulation of the summation symbol and indices. We have opted to display the proof as a separate boxed item (Box 1).

  It is important to keep in mind that "doing mathematics" involves a great many things: experimentation and generating data; studying the data for patterns, using all the techniques one possesses; conjecturing and hypothesizing; systematically checking out our conjectures and hypotheses; and, finally, proving our conjectures and hypotheses. Sometimes, one or more of these steps requires a substantial amount of algebra and symbol manipulation. *All these are part and parcel of the subject.*

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Proof of relation (5)

We are required to prove the following relation:

$$f_k(n) = \sum_{r=1}^{k+1} \binom{n}{r} \cdot T(k,r),$$

where $f_k(n) = 1^k + 2^k + \cdots + n^k$. We have, from (3):

$$\begin{aligned}
f_k(n) &= \sum_{r=1}^{k} r! \cdot \binom{n+1}{r+1} \cdot S(k,r) = \sum_{r=1}^{k} r! \cdot \left(\binom{n}{r} + \binom{n}{r+1} \right) \cdot S(k,r) \\
&= \sum_{r=1}^{k} r! \cdot \frac{T(k-1,r)}{(r-1)!} \cdot \left(\binom{n}{r} + \binom{n}{r+1} \right) \\
&= \sum_{r=1}^{k} r \cdot T(k-1,r) \cdot \binom{n}{r} + \sum_{r=1}^{k} r \cdot T(k-1,r) \cdot \binom{n}{r+1} \\
&= \sum_{r=1}^{k} \Big(T(k,r) - (r-1) \cdot T(k-1,r-1) \Big) \cdot \binom{n}{r} + \sum_{r=1}^{k} r \cdot T(k-1,r) \cdot \binom{n}{r+1} \\
&= \sum_{r=1}^{k} T(k,r) \cdot \binom{n}{r} - \sum_{r=1}^{k} (r-1) \cdot T(k-1,r-1) \cdot \binom{n}{r} + \sum_{r=1}^{k} r \cdot T(k-1,r) \cdot \binom{n}{r+1} \\
&= \sum_{r=1}^{k} T(k,r) \cdot \binom{n}{r} - \sum_{r=0}^{k-1} r \cdot T(k-1,r) \cdot \binom{n}{r+1} + \sum_{r=1}^{k} r \cdot T(k-1,r) \cdot \binom{n}{r+1} \\
&= \sum_{r=1}^{k} T(k,r) \cdot \binom{n}{r} + k \cdot T(k-1,k) \cdot \binom{n}{k+1} \\
&= \sum_{r=1}^{k} T(k,r) \cdot \binom{n}{r} + k \cdot (k-1)! \cdot \binom{n}{k+1} = \sum_{r=1}^{k} T(k,r) \cdot \binom{n}{r} + k! \cdot \binom{n}{k+1} \\
&= \sum_{r=1}^{k+1} T(k,r) \cdot \binom{n}{r}.
\end{aligned}$$

Box 1. Details of the proof of relation (5)