ClassRoom

The 'CONJOINING' error in SCHOOL ALGEBRA

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Many students who become reasonably proficient in arithmetic face great difficulties with school algebra, which may lead to a cascading spiral of low performance and eventually to their giving up mathematics. Understanding why these difficulties occur is the first step in changing one's teaching to deal effectively with them. If students can be helped through these difficulties, it can lead to fewer students dropping out of mathematics or out of learning tracks like science that need mathematics. This article briefly discusses a particular error in algebra that is quite common and the possible reasons for this error. Even though our focus is just on one particular error, analysing the error leads to insights about deeper issues that students have with algebra. Our discussion draws on research in mathematics education on the learning of algebra. Such research, done in many places across the world including in India, not only analyses various kinds of errors that students make but also develops better approaches to the teaching of algebra. A discussion on this research would be too long to include here. However, we provide references to articles that describe one such approach to the teaching of algebra that was developed at the Homi Bhabha Centre.

The Conjoining Error

The conjoining error, which is quite common, is seen in responses to the task of simplifying algebraic expressions.

$$1)\ 5 + 2a = 7a\ * \qquad 2)\ 5a + 2b = 7ab\ *$$

(The asterisk to the right indicates that the response is incorrect.)

Many teachers would recognize this as one of the most frequent errors that they see in students' work. The name "conjoining

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error" refers to the incorrect joining of the two terms. Before we proceed, we need to clarify that both sentences are actually *not* incorrect if they are interpreted as equations rather than identities. The sentence (1), interpreted as an equation, would be true when $a = 1$. Similarly sentence (2) would be true for all pairs of a and b given by the function $b = 5a/(7a - 2)$ *with* $a \neq 2/7$. The sentences are false only when they are interpreted as identities, i.e., a sentence that is true for all values of the variables. A *simplification* of the expression on the left to the one on the right is possible only if the LHS is identically equal to the RHS.

How would you deal with this error if you were a teacher? A frequent suggestion is that we must stress the concept of 'like' terms. You can add like terms just as you can add apples to apples but not apples to bananas. But it is very hard to remember this, especially when faced with (2) above. A student might think that we can always put 5 apples and 2 bananas together in a basket to get 7 fruits, which are apples and bananas. Thus the student responds with $5a + 2b = 7ab$ *, and the 'fruit salad' algebra breaks down. Moreover, the student is led to think that '*a*' in the expression stands for things like apples, rather than standing for a number, which is an even more serious misunderstanding.

Another way to think about the error is to see if there is a counterpart in arithmetic. We can indeed find a counterpart to (1), which is presented below in (1a), but it is difficult to think of a counterpart for (2).

1) $5 + 2a = 7a$ *

1*a*) $5 + 2 \times 3 = 7 \times 3 = 21$ *

The error in (1a) is that the convention for the order of operations or the BODMAS rule has been broken. Since the error in the algebraic sentence looks very similar to the arithmetic error, one may think that the right way of dealing with this is to remind students of the BODMAS rule and give them practice in applying it. However, the underlying reasons for the errors in (1) and (1a) may be very different, as we will see. In other words, although the error in (1) and (1a) are *mathematically* similar, the *cognitive* aspects that lead to the errors may be very different.

First, let us think about why the rule of order of operations is taught. Many mathematicians might say that such a rule is unnecessary. In fact, the LHS expression in (1a) is ambiguous because we have not put brackets. We can put brackets in two ways to get two different values:

$$(5 + 2) \times 3 = 7 \times 3 = 21 \quad \text{or} \quad 5 + (2 \times 3) = 5 + 6 = 11$$

Once we put brackets, both of these are correct. So a mathematically correct view would be that we must put brackets whenever we have two or more binary operations in a single expression. Otherwise, the expression would be ambiguous, except when the binary operations are all addition or all multiplication, in which case the different ways of putting brackets lead to the same result.

$$(5 + 2) + 3 = 5 + (2 + 3) = 10 \quad \text{and}$$
$$(5 \times 2) \times 3 = 5 \times (2 \times 3) = 30$$

This, of course, is because of the associative property of addition and multiplication.

In the light of the above, it is clear that the BODMAS or any other rule for the order of operations is a convention that allows us to interpret an expression with multiple binary operations *when brackets are not written*. (Incidentally, the BODMAS rule when applied precisely to an expression like 30 – 10 + 10, leads to an error since it suggests that addition is to be done before subtraction. 'BODMSA' is more faithful to the convention than 'BODMAS'.) Why do we need such a convention? Why not simply put brackets for all numerical expressions and do away with the need for such rules, which anyway are difficult for students to remember and apply correctly? It is worth thinking about this proposal.

It appears that the actual reason for teaching a rule like BODMAS is that it prepares students to interpret and work with algebraic expressions. The rules for simplifying and manipulating algebraic expressions will break down if the expressions do not have a definite and unambiguous value when the variables are substituted with numbers.

What about the suggestion to put lots of brackets to make the expression unambiguous? Putting brackets in algebraic expressions makes them hard to read and interpret and so we must minimize the use of brackets. Of course, the BODMAS rule also applies to algebraic expressions and multiplication gets priority over addition and subtraction, just as in numerical expressions. However, the BODMAS rule is rarely invoked while parsing algebraic expressions. This is because algebraic expressions use visuospatial and reading conventions to encode which operations get priority. For example, the sign for the multiplication operation is omitted both in writing (and reading) algebraic expressions to signal the priority of multiplication over addition or subtraction. The convention for writing (and reading) exponents signals the priority of this operation over others. Note that these conventions are very different from the BODMAS convention for numerical expressions. Hence it is unlikely that violating the BODMAS rule, which underlies the error in (1a) above, also causes the error in (1). Further, even if a student knows the BODMAS rule perfectly well, she or he cannot use the rule to simplify the expression in (1) because of the presence of the letter variable instead of a number. Thus, the presence of letter variables constrains the use of rules of order of operations and limits their usefulness.

There is also a noticeable difference between the BODMAS convention and the conventions for algebraic expressions. The former is a verbal rule that states which operations precede which and is encoded through an abbreviation or a mnemonic. In contrast, the conventions for algebraic expressions are visuospatial and based on ways of writing. It suggests that working with the BODMAS rule may not be of great help in learning to parse algebraic expressions correctly. Is it possible to work with numerical and algebraic expressions using similar conventions for parsing both kinds of expressions? Indeed, an approach to working with numerical expressions that uses visual parsing of "terms" supported by appropriate naming has been found to be helpful in bridging the gap between arithmetic and algebra (Banerjee & Subramaniam, 2012).

Let us now return to the conjoining error. One of the explanations proposed for why students make this error is that they find "unclosed expressions" unacceptable as answers. In other words, their experiences in arithmetic leads them to think that a simpler looking expression needs to be written to the right of the "=" sign. A response such as the following

$$5 + 2a = 5 + 2a \quad \text{or} \quad 5a + 2b = 5a + 2b$$

may appear as a kind of cheating – writing the question again as the answer because the unclosed expression on the right " $5 + 2a$ " looks like a question rather than an answer. In contrast, the "closed" expression "$7a$" looks compact and like an answer. The reluctance to accept unclosed expressions as answers brings to the fore a move that is at the heart of algebraic thinking, which is to actually accept a question as an answer! That is, we allow the expression that shows the operations to be carried out also to stand for the result of the operation. Thus $5 + 2$ or $2 \times 3 + 1$ are not only expressions that tell us to carry out certain operations, but may also stand for the answer that is obtained as the result, namely, the number 7. Thus $5 + 2a$ and $7a$ stand for numbers, and these numbers are in general not equal when the same value of a is used in both expressions. Seeing this depends on seeing that the sentence in (1) or (2) is about the equality of numbers on the left and right sides of the "=" sign, numbers which are "variable" and become "fixed" once the variables a and b are substituted. Indeed, many students interpret the "=" sign as asking one to "do something and write the answer" rather than as stating the equality of expressions on either side of the sign. This interpretation leads to errors when faced with a question like "11 + 7 = __ + 9" to which students may respond with "11 + 7 = **18** + 9 = 27*". Such students also think that there is something very wrong with a sentence like "3 = 3".

The reification principle

The algebraic principle of allowing an expression to stand for the result of the expression, sometimes called the "reification principle," is already used in arithmetic. For example, the result of dividing 5 by 7 is expressed as 5/7. In general, $a \div b = a/b$, which is very much like "writing the question as the answer". Similarly the square root of 2 (or any number a) is simply written as $\sqrt{2}$ (or $\sqrt{a}$ in general). This is a very important aspect of algebra that students generally miss out on and it helps enormously if teachers point it out to them. Understanding the reification principle allows them, for example, to recognize that $(a+b)^2$ and $a^2 + 2ab + b^2$ are different expressions for the same number (when a and b are substituted with numbers). It allows them to read and grasp expressions, because they contain information about the number represented. Thus, 48 + 47, 45 + 45 + 3 + 2, and 50 + 50 – 2 – 3 are different expressions for the same number and hence equivalent, but encode different ways in which the number 95 is 'composed' from other numbers. (We have described the information encoded by an expression as the *operational composition* of a number in Subramaniam and Banerjee, 2011.) An expression such as 199 + 70 × 0.5, written to show how a cellphone tariff is calculated, suggests that there is a fixed cost of Rs 199 and a rate of Rs 0.5 per minute. Even the numeral "536", is a short form of the expression 5 × 100 + 3 × 10 + 6, which shows the operational composition of the number 536 in terms of multi-units which are different powers of ten. The capability to "read" expressions in this manner is important in algebra and requires the understanding of the reification principle as its foundation. Again we refer to Banerjee and Subramaniam (2012) for a way in which such ability can be developed while working with numerical expressions as a preparation for algebra.

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