ClassRoom

Visual Proof of THE TWO-VARIABLE AM-GM INEQUALITY

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The AM-GM inequality which is normally stated in the following form: *"If a and b are any two non-negative real numbers, then*

$$\frac{a+b}{2} \geq \sqrt{ab},$$

with equality holding if and only if a = b," may be stated in the following equivalent form, where we have used the numbers a^2 and b^2 rather than a and b: For any two positive real numbers a and b, we have

$$\frac{a^2+b^2}{2} \geq ab, \qquad (1)$$

with equality holding if and only if $a = b$.

The algebraic proof of (1) consists of recognising that $a^2 + b^2 - 2ab$ is a perfect square, namely, $(a - b)^2$ which is always non-negative as a and b are real numbers. This is probably the simplest and the most direct proof. But to justify the word 'geometric' in the definition of the 'geometric mean' and hence in the name of the inequality, it is desirable to have a geometric proof. One such proof has appeared on pp.42–43 of *At Right Angles*, August 2017, in an article by Shailesh Shirali. In addition to proving the inequality, the proof also gives a geometric construction for the geometric mean.

But if merely proving (1) geometrically is the goal, there is a much more direct proof which we now give.

Without loss of generality, assume $a \geq b$. Construct right-angled isosceles triangles OAA' and OBB' with legs a and b respectively, with B' lying on OA' as shown in Figure 1. Extend BB' to meet AA' at C.

Keywords: AM-GM inequality, visual proof

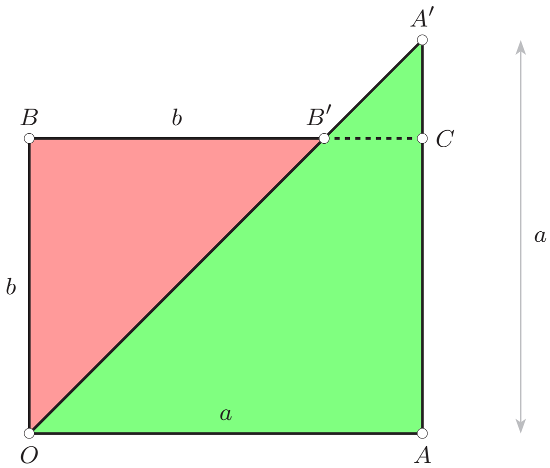

Figure 1

It is clear that the union of the two triangles OAA' and OBB' covers the rectangle $OACB$ and hence has a higher area except when $B' \equiv A'$. The inequality (1) follows by taking the areas of these two triangles and of the rectangle $OACB$.

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