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Solution of Ramanujan's DOOR NUMBER PROBLEM by using VARGAPRAKRITI

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The Strand Magazine Door Number Problem, now eternally associated with Ramanujan, may be stated more generally as follows: *A street has n houses numbered consecutively, the numbers starting from 1, and there is a house numbered x such that the sums of the house numbers on each side of x are the same. Find n and x, given that n lies within some specified range.*

As per the stated information, $1 + 2 + \cdots + (x - 1) = (x + 1) + (x + 2) + \cdots + n$. Add $1 + 2 + \cdots + (x - 1) + x$ to both sides:

$$2(1 + 2 + \cdots + (x - 1)) + x = 1 + 2 + \cdots + n.$$

The expression on the left side simplifies to $x(x - 1) + x = x^2$. Hence:

$$x^2 = \frac{n(n+1)}{2}$$

To solve this equation, Ramanujan used continued fractions. His approach has become part of history ever since!

Our question is: *Is there an approach apart from continued fractions?* The answer is: **Yes**. I demonstrate such an approach. First I rearrange the above equation into a more familiar shape. Multiply both sides by 8 and then add 1 to both sides; I get:

$$8x^2 + 1 = 4n(n + 1) + 1,$$

i.e.,

$$(2n + 1)^2 - 8x^2 = 1.$$

This is nothing but an instance of the Indian mathematician Brahmagupta's *Vargaprakriti*, the indeterminate quadratic equation $y^2 - kx^2 = 1$ on which he worked in the 6th century CE.

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Here it is of the form $y^2 - 8x^2 = 1$, where $y = 2n + 1$. (See Box 1 for the history of this equation.)

Therefore, by finding the solutions (which are infinite in number) of $y^2 - 8x^2 = 1$, I should get solutions to the generalised Strand Magazine Door Number Problem.

According to the composition law (***bhāvanā***) found by Brahmagupta, if (y_1, x_1) and (y_2, x_2) are solutions to $y^2 - kx^2 = 1$, then so is $(y_1y_2 + kx_1x_2, y_1x_2 + x_1y_2)$. By applying *bhāvanā* again and again, infinitely many solutions can be generated.

The most obvious solution of $y^2 - 8x^2 = 1$ is $y = 3$, $x = 1$, i.e., $n = 1$, $x = 1$. This solution corresponds to there being just one house in the street.

Starting with the pair (3, 1) and applying *bhāvanā* on itself, I get the solution

$$(3^2 + 8 \cdot 1^2, 2 \cdot 3 \cdot 1) = (17,6),$$

i.e., $n = 8$, $x = 6$, or 8 houses; the desired one is the 6th one. Next, applying *bhāvanā* on the pairs (17,6) and (3,1), I get the solution

$$(17 \cdot 3 + 8 \cdot 6 \cdot 1, 17 \cdot 1 + 6 \cdot 3) = (99, 35),$$

i.e., $n = 49$, $x = 35$, or 49 houses; the desired one is the 35th one. Thus I generate infinitely many solutions of $y^2 - 8x^2 = 1$ and find the answer according to the specified range of n. Some pairs of solutions obtained in this manner and the corresponding values of n are listed in the table below.

$y = 2n + 1$	x (desired Door Number)	n (no. of houses)
3	1	1
17	6	8
99	35	49
577	204	288

The Strand Magazine problem states that $50 \le n \le 500$, hence $n = 288$, $x = 204$. *So there are 288 houses, and the desired house number is 204.*

Note from the editors: Another method for solving this problem has been described in the article on Ramanujan by Utpal Mukhopadhyay, elsewhere in this issue.

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Vargaprakriti

Vargaprakriti refers to the second-degree indeterminate equation in two variables, $y^2 - Nx^2 = 1$, which must be solved over the positive integers. Here N is an arbitrary positive integer. For example, the equation that arises in the door number problem is $y^2 - 8x^2 = 1$.

Literally, Vargaprakriti means 'equation of the multiplied square'; Varga means 'coefficient' and refers to the number N in the equation $y^2 - Nx^2 = 1$.

The equation is better known as the ***'Pell equation'*** (after John Pell, a 17th century English scholar), but the name is now known to be a historical inaccuracy. These equations were first studied in detail by Brahmagupta in the 6th century CE, and later by Bhaskaracharya II (who developed a so-called *'Chakravala'* or cyclic method of solution) and others. A more appropriate name for the equation would therefore be the Brahmagupta-Bhaskara equation.

For more on this topic, the reader is directed to the following excellent reference material which we have used freely: http://www-groups.dcs.st-and.ac.uk/history/Miscellaneous/Pearce/Lectures/Ch8_6.html