Using Co-Ordinate Geometry to find the GCD and LCM of TWO NUMBERS *a* & *b*

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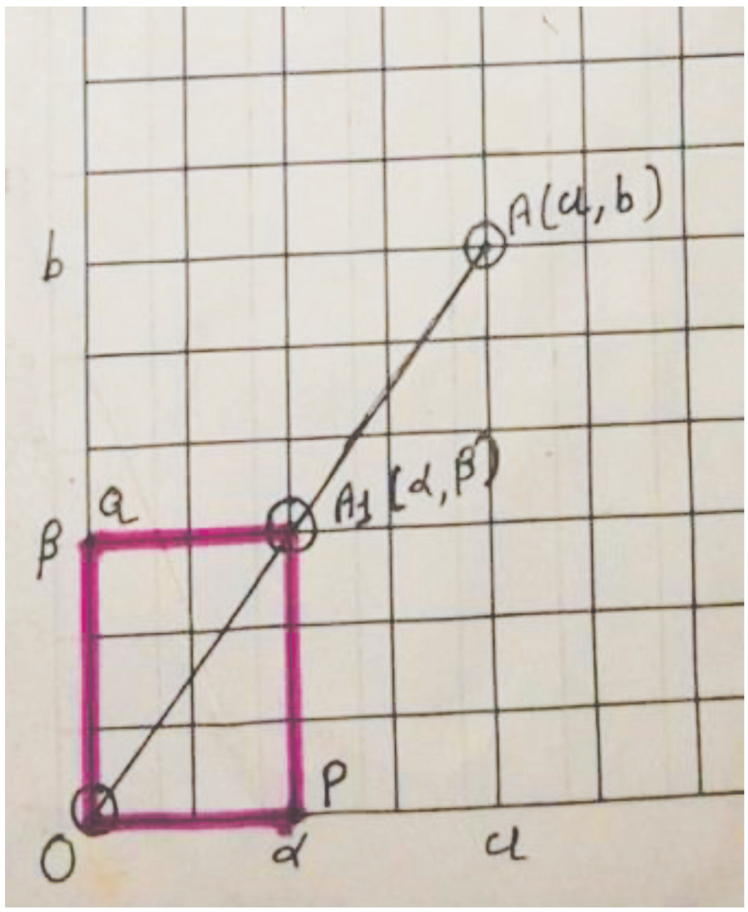

Figure 1.

Claim: Let ***a, b*** be natural numbers. If we plot the point (a, b) on a square grid and draw the line joining $(0, 0)$ to (a, b), then the GCD of a and b is given by the number of grid points on this line decreased by 1.

Keywords: Natural Numbers, LCM, HCF, Coordinate Geometry, Lattice Points, Visualization, Representation

Illustration: Find the GCD of 4 and 6.

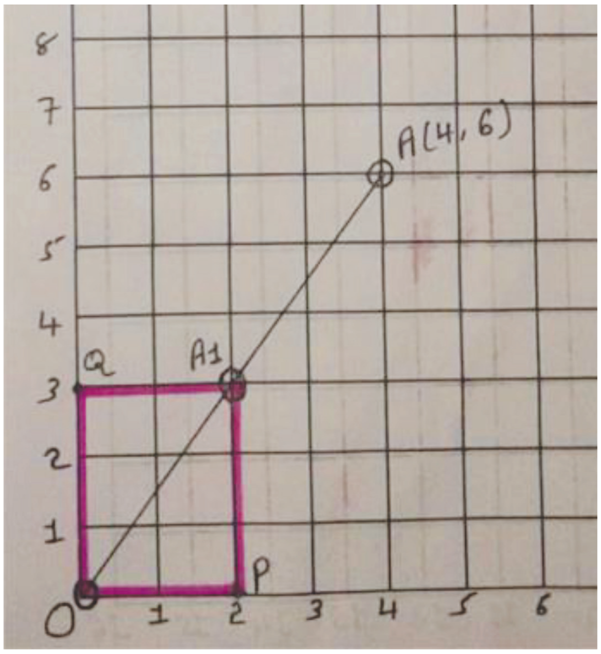

Figure 2

- Represent point A (4, 6) in the sheet.
- Construct line segment $\overline{OA}$ whose end points are O (0, 0) and A (4, 6).
- Find the number of lattice points on $\overline{OA}$. Here, the number is 3.
- **GCD = number of lattice points – 1 = 3 – 1 = 2.**

  Thus, GCD of 4 and 6 is 2.

Claim: In Figure 1, choose the lattice point O (0, 0) and the lattice points next to it on the same line, say, A_1, construct the rectangle OPA_1Q, and then find the number of squares contained in the rectangle OPA_1Q. Then:

LCM (*a, b*) = GCD (*a, b*) × number of squares contained in Rectangle OPA$_1$Q.

Illustration: Find the LCM of 4 and 6

- In Figure 2, choose lattice point O (0, 0) and the 'next' lattice point, A_1.
- Construct rectangle OPA_1Q. Find the number of squares in OPA_1Q.
- The LCM can be found with the help of the following formula

LCM = GCD × number of squares contained in rectangle OPA_1Q = 2 × 6 =12.

Explanation

1. Consider point A (a, b) in the sheet (Figure 1). Write the equation of $\overleftrightarrow{OA}$ as follows.

   $$\therefore \overleftrightarrow{OA} : y = \frac{b}{a}x$$

If k is a common divisor of a and b, then $a = k\alpha$ and $b = k\beta$ for some positive integers α and β.

If we substitute $x = \alpha$ in the equation of $\overleftrightarrow{OA}$, we get $y = \beta$ and hence point $A_1(\alpha,\beta)$ is on line segment $\overline{OA}$.

The equation of $\overleftrightarrow{OA}$ can therefore be simplified to $y = \frac{\beta}{\alpha}x$.

We see that y has an integer solution only if x is a multiple of α.

Now $0 \le x \le a$, therefore $0 \le x \le k\alpha$. There are k + 1 multiples of α starting with 0 and ending with $k\alpha$.

$\therefore$ **Number of lattice points = k + 1**

$\therefore$ **k = number of lattice points – 1**

$\therefore$ **GCD (*a,b*) = number of lattice points – 1**

2. Now LCM (a,b) = LCM$(k \cdot \alpha,\ k \cdot \beta)$

   $\therefore$ LCM (a,b) = k · LCM(α,β) where α *and* β are relatively prime.

   $\therefore$ LCM$(a,b) = k \cdot \alpha\beta$

   $\therefore$ **LCM (a,b) = GCD (a,b) × Area of rectangle OPA_1Q.**

   $\therefore$ **LCM (a,b) = GCD (a,b) × number of squares in rectangle OPA_1Q.**

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