ClassRoom

On DIVISIBILITY BY 27

SHAILESH SHIRALI

Notation. We denote the sum of the digits of a positive integer n by $\mathrm{SD}(n)$. The notation $a \mid b$ means: 'a is a divisor of b', i.e., 'b is a multiple of a.' Throughout, we work in base 10.

Two well-known statements. The following two statements are very well-known:

(1) A positive integer n is divisible by 3 if and only if $\mathrm{SD}(n)$ is divisible by 3.

(2) A positive integer n is divisible by 9 if and only if $\mathrm{SD}(n)$ is divisible by 9.

Or, more compactly: $3|n \Leftrightarrow 3|SD(n)$, $9|n \Leftrightarrow 9|SD(n)$. Both the statements are true and easy to prove. Examining them, one may be tempted to generalise:

$\boxed{27|n \Leftrightarrow 27|\mathrm{SD}(n).}$??? Status unclear ???

But is this claim true? Note that it consists of two sub-claims: *For any positive integer n,*

(1) *If n is divisible by 27, then* $\mathrm{SD}(n)$ *is divisible by 27;*

(2) *If* $\mathrm{SD}(n)$ *is divisible by 27, then n is divisible by 27.*

How do we check whether either of these claims is true?

Notion of a counterexample. A standard way of proceeding when confronted with a statement about which we are uncertain (regarding whether it is true or false) is to actively look for a

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counterexample; i.e., a situation where the given statement is falsified. If we find such a counterexample, then the given statement must be false.

Illustrations. Here are some illustrations of this notion.

(1) Claim: *If n is a positive integer, then* $4n^2+1$ *is a prime number*. This is true for $n = 1,2,3$ but false for $n = 4$. So $n = 4$ provides a counterexample to the stated claim.

(2) Claim: *If p is a prime number, then* 2^p-1 *is a prime number*. This is true for $p = 2,3,5,7$ but false for $p = 11$. So $p = 11$ provides a counterexample to the stated claim.

Finding a counterexample is clearly a very effective way of disposing off false claims. It is an extremely important notion in the study and exploration of mathematics.

Looking for a counterexample. Armed with this notion, let us look for a counterexample to the statement *"If a positive integer n is divisible by* 27, *then* SD(n) *is divisible by* 27."

Once we set ourselves this task, the question becomes absurdly simple to answer. Indeed, the very first (positive) multiple of 27 (namely, 27 itself) is a counterexample! For, 27 is a multiple of 27. On the other hand, SD(27) = 9, which is not a multiple of 27. *So the statement under study is not true in general.* (That got resolved rather quickly, didn't it? Too quickly, perhaps?)

What about the converse: "*If* SD(n) *is divisible by* 27, *then n is divisible by* 27"? Is this true?

What combinations of digits yield a sum of 27? The simplest such combination is 9 + 9 + 9= 27; and we find that 999 is a multiple of 27; indeed, 999 = 27 × 37. Next in simplicity we have the combination 1 + 8 + 9 + 9 = 27. And 1899 turns out to be not divisible by 27! Indeed, 1899 = (70 × 27) + 9, so 1899 ≡ 9 (mod 27). *So the condition that* SD(n) *is divisible by* 27 *is not enough to force n to be divisible by* 27.

The counterexample just found may have been found a little too easily (it is rather disappointing when things happen too easily, isn't it?); we may wonder whether we could have proceeded in a more systematic way. Indeed we can. Having seen the number 999 above, our mind may naturally turn to the number 888. This number is divisible by 3 but not by 9 and therefore not by 27 either. We find by actual division that 888 ≡ 24 (mod 27) ≡ −3 (mod 27). On multiplication by 10, we obtain:

$$8880 \equiv -30 \pmod{27} \equiv -3 \pmod{27}.$$

We infer that the numbers 888, 8880, 88800, . . . all leave the same remainder (namely, 24) under division by 27. Continuing, we infer that the numbers 8883, 88803, 88830, 888030 . . . are all divisible by 27. Note that each of these numbers has sum-of-digits equal to 27. And since $111 \equiv 3 \pmod{27}$, exactly the same statement can be made for the numbers 888111, 8880111, 8881110, 88811100 . . . : each of them is divisible by 27, and each of them has sum-of-digits equal to 27.

On the other hand, note that 1011 ≡ 12 (mod 27) and 1101 ≡ 21 (mod 27). (Please check these computations.) So 1011 and 1101 do not leave remainder 3 under division by 27. It follows that the numbers 8881011 and 8881101 are *not* divisible by 27 (on division by 27, they leave remainders of 9 and 18, respectively). However, each of them has sum-of-digits equal to 27. Therefore, each of these numbers contradicts the claim that if SD(n) is divisible by 27, then n is divisible by 27.

More counterexamples can be generated by arguing in this manner. It may be a good exercise for you to do so.

What we have found is that both the statements under study are false, and in both cases, we have determined that this is so by finding counterexamples. Our conjecture has thus met with a sorry end!

In much the same way, we can put to rest the following claim,

$$\text{If } 81 \mid \mathrm{SD}(n), \text{ then } 81 \mid n,$$

just in case anyone ever made such a claim. But we shall leave the task of finding the counterexample to you. (See [1] for more on this.)

Is there any test for divisibility by 27?

After the disappointing experience above, we may wonder whether there is any worthwhile test for divisibility by 27. There is, and it is provided by the observation that 999 is divisible by 27 (check: $999 = 27 \times 37$, remainder 0). From this, it follows that $1000 \equiv 1 \pmod{27}$.

This observation gives rise to the following test of divisibility. Let n be the given positive integer. We assume to start with that $n \geq 1000$. Let b denote the number formed by the last three digits of n, and let a denote the 'rest' of the number (with those three digits deleted); so $n = 1000a + b$. For example, if $n = 123456$, then $b = 456$ and $a = 123$. We now replace n by $n_1 = a + b$ and continue the computations with n_1 in place of n. It is not difficult to see that if n has four or more digits, then n_1 is substantially smaller than n. The crucial fact now is: $n \equiv n_1 \pmod{27}$. On this observation rests the algorithm.

Continuing in this manner, we ultimately obtain a number with three or fewer digits. There are two ways to proceed at this stage. The first is based on recognition: we assume that we are able to *recognise* all three-digit multiples of 27. (This is not as difficult as it sounds. I'm sure that we can all rise to the challenge!)

The other approach, which we use if n has three or fewer digits, uses the digits of n. It is based on the observation that $27 \times 4 = 108 = 100 + 8$, and therefore that $100 \equiv -8 \pmod{27}$.

Let a be the hundreds digit of n and let b be the two-digit number formed by the remaining two digits (i.e., the tens digit and the units digit) of n. Note that this means that $n = 100a + b$. For example, if $n = 453$, then $a = 4$ and $b = 53$. Now replace n by the number $n_1 = b - 8a$. *Then n is divisible by 27 if and only if n_1 is divisible by 27.* Observe that at the end, we again fall back on recognition: we assume that we are able to recognise all the two-digit multiples of 27. But that, surely, is not asking for too much!

Examples. In the examples shown below, we use the symbol $\rightsquigarrow$ to denote the following number replacement operation:

- If n has four or more digits and $n = 1000a + b$ where b has three or fewer digits $(\text{i.e., } b < 100)$, then $n \rightsquigarrow a + b$.
- If n has three or fewer digits and $n = 1000a + b$ where b has two or fewer digits $(\text{i.e., } b < 100)$, then $n \rightsquigarrow b - 8a$.

(1) $n = 123456$. We now obtain:

$$123456 \rightsquigarrow 123 + 456 = 579.$$

Next:

$$579 \rightsquigarrow 79 - (8 \times 5) = 79 - 40 = 39.$$

Since 39 is not a multiple of 27, we conclude that 123456 is not a multiple of 27.

(2) $n = 11134233$. Then we have:

$$\begin{aligned} 11134233 &\rightsquigarrow 11134 + 233 = 11367 \\ &\rightsquigarrow 11 + 367 = 378 \\ &\rightsquigarrow 78 - (8 \times 3) = 78 - 24 = 54. \end{aligned}$$

Since 54 is a multiple of 27, we conclude that 11134233 is a multiple of 27.

We now offer a formal proof that this algorithm works correctly.

Proof of correctness of algorithm. Let a and b be any two integers, and let n and n_1 be defined as follows:

$$n = 1000a + b, \; n_1 = a + b.$$

Then we observe that:

$$n - n_1 = 999a = 27 \times 37a,$$

i.e., $n - n_1 \equiv 0 \pmod{27}$ Since the difference between n and n_1 is a multiple of 27, if either of them is a multiple of 27, so must be the other one.

Similarly, if n and n_1 are defined as follows:

$$n = 100a + b, \; n_1 = b - 8a,$$

then we observe that:

$$n - n_1 = 108a = 27 \times 4a,$$

i.e., $n - n_1 \equiv 0 \pmod{27}$. Since the difference between n and n_1 is a multiple of 27, if either of them is a multiple of 27, so must be the other one. The conclusion here is identical to the earlier one.

So the replacement of n by n_1 does not alter the status of divisibility by 27. This proves the correctness of the algorithm. But more can be said. Since in each case we get $n - n_1 \equiv 0 \pmod{27}$, it follows that n and n_1 *leave the same remainder under division by* 27. So this algorithm also provides a way of computing the remainder when we divide a large integer by 27.

References

1. Alexander Bogomolny, "Division by 81 from Interactive Mathematics Miscellany and Puzzles", https://www.cut-the-knot.org/Generalization/81.shtml, Accessed 17 November 2017

SHAILESH SHIRALI is Director of Sahyadri School (KFI), Pune, and Head of the Community Mathematics Centre in Sahyadri School/Rishi Valley School (AP). He has been closely involved with the Math Olympiad movement in India. He is the author of many mathematics books for high school students, and serves as Chief Editor for A*t Right Angles*. He may be contacted at shailesh.shirali@gmail.com.