Low Floor High Ceiling Tasks

Here we go round…

NEWER AND NEWER SPIRALS - OPEN AND SHUT CASES

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We continue our Low Floor High Ceiling series in which an activity is chosen – it starts by assigning simple age-appropriate tasks which can be attempted by all the students in the classroom. The complexity of the tasks builds up as the activity proceeds so that students are pushed to their limits as they attempt their work. There is enough work for all, but as the level gets higher, fewer students are able to complete the tasks. The point, however, is that all students are engaged and all of them are able to accomplish at least a part of the whole task.

In the March 2017 issue, Khushboo Awasthi had described an investigation of the familiar Square Root Spiral, which had taken her along unexpected paths filled with mathematical discoveries. At the end of the article, she posed some questions for the reader to investigate and we did just that! We share our bonanza of findings with you, and as usual, the tasks are arranged from Low Floor to High Ceiling. This time, we include some investigations with the free dynamic geometry software GeoGebra; regular constructions with compass and ruler will do the job just as well!

This is what the Square Root Spiral looks like (see Figure 1).

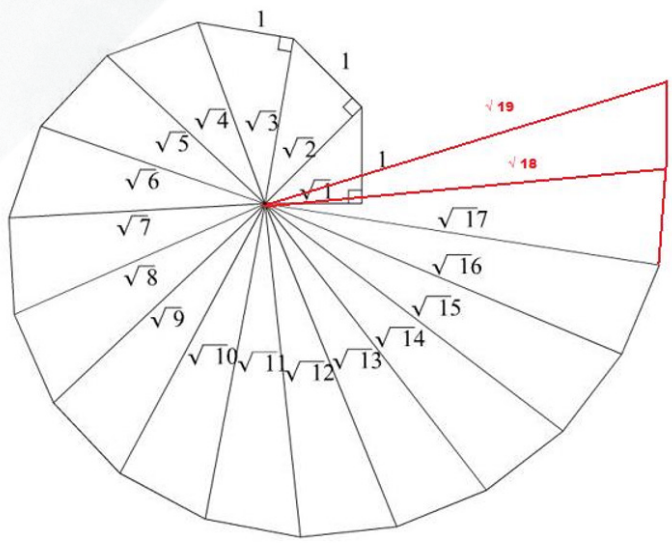

Figure 1: The Square Root Spiral

Keywords: Right-angled triangles, Pythagoras, similarity, reflections, construction

The question posed at the end of the article was this:

What will happen to the spiral if, at every iteration, we vary the length of the opposite side and make it equal to the base?

Let's explore!

Steps to create Square Root Spiral Version 2.0

1. On a blank sheet of paper, draw a line segment AB, of unit length, in the middle of the paper. At point B, construct a perpendicular line segment of unit length (same as AB), named BC. The hypotenuse AC will hence be of length$\sqrt{2}$. (Refer Figure 2).

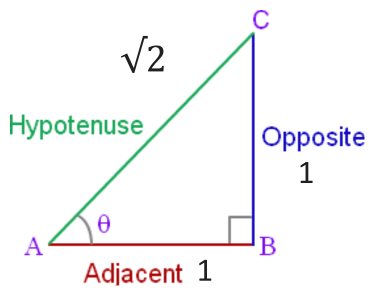

Figure 2

2. Construct a line at point C perpendicular to segment AC. Construct line segment CD of length $\sqrt{2}$ (same as AC) on this line. Then, draw AD to form the hypotenuse with AC as base and CD as the opposite side of the new right angled triangle. So, **AD** = $\sqrt{4}$ = 2.

3. Similarly, construct a line segment DE of length AD = 2, from point D perpendicular to AD. Join AE to form the hypotenuse of the right angled triangle with AD as base.

Repeat this process to get more right angled triangles. The only point to remember is that the perpendicular side of the new triangle has to be the same length as the hypotenuse of the previous triangle. (Ref. Figure 3).

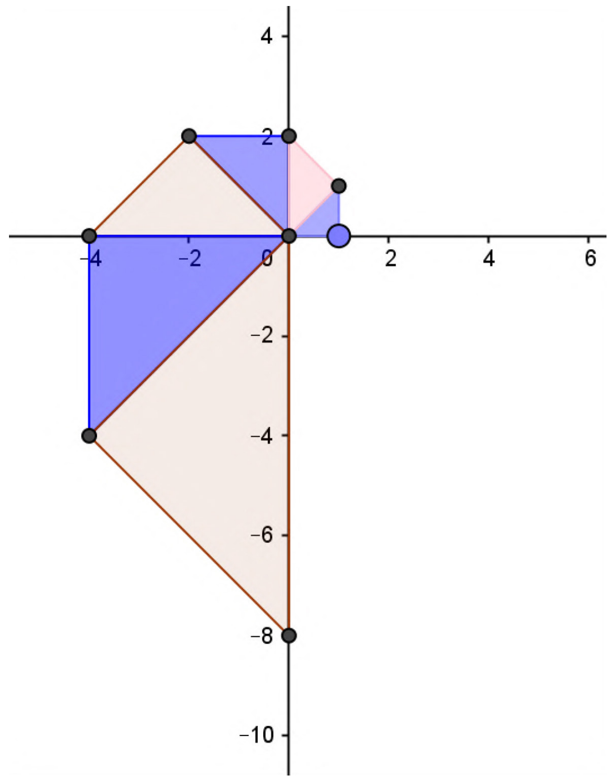

Figure 3

Task 1

Briefly describe the characteristics that are common to each of the triangles.

Teacher's Notes: The low floor start simply requires students to recognize that the triangles are all similar, being right angled isosceles triangles. Some of the students may recognize that if the base and height are both a, then the hypotenuse of the triangle is $\sqrt{2}$a. This is a chance for the teacher to point out triangles that are similar but not congruent.

Task 2

How are the triangles related to each other? Compare the lengths of the sides and the areas.

Teacher's Notes: Building on the previous question, the teacher can facilitate the students to record their findings and recognize patterns in a table such as the one given below:

No. of Triangle (n)	Base (b)	Opposite side (a)	Hypotenuse (c)	Area
1	1	1	$\sqrt{2}$	½
2	$\sqrt{2}$	$\sqrt{2}$	2	1
3	2	2	$\sqrt{8}$	2
4	$\sqrt{8}$	$\sqrt{8}$	4	4
5	4	…..	…..	
n	…...	……	…..	

Table 1: Length of base, opposite side and hypotenuse

Clearly, the sides of each new triangle are $\sqrt{2}$ times the sides of the old triangle and the area is twice the area of the previous triangle.

Task 3

Does the spiral close in this case? Justify your stance. Describe what happens in the second loop of the spiral.

How do the sides and area of the first triangle in the second loop compare with those of the first triangle in the first loop? What about succeeding loops? Does any pattern emerge?

Teacher's Notes: As usual, we ramp up the difficulty level with the need to *justify*. As students construct the triangles, they will realise that the spiral closes simply because the angle at the centre of the spiral is 45° in each case and so after 8 triangles the angle adds up to 360° and the spiral closes. Communicating this in words or writing is, however, a skill that students may need to practise.

Students may complete the table to obtain the base and height of the 9^{th} triangle (the first in the second loop of the spiral) to be 16, the hypotenuse to be $16\sqrt{2}$ and the area to be 128. So the sides are 16 (= 2^4) times the sides of the first triangle in the first loop and the area is 256 (= 2^8) times the area of this triangle. A great chance to see that the ratio of the areas is the square of the ratio of the sides. And a chance to reiterate and see that similarity of triangles means that the corresponding angles are the <u>same</u> but the sides are in the <u>same ratio</u>. If a table is made of the ratio "the side of the first triangle in n^{th} loop : the side of the first triangle", the following pattern emerges: $1, 2^4, (2^4)^2, (2^4)^3 \ldots (2^4)^{(n-1)}$. A similar pattern arises for the areas.

Task 4

These triangles proceed in the anti-clockwise direction. Construct the triangles which proceed in the clockwise direction, with each new height being equal to the previous hypotenuse.

Conjecture about what would happen to the ratio of the areas if the triangles proceeded clockwise.

Teacher's Notes: Construction of these triangles is a more difficult task requiring the understanding that the right angle is now opposite the base of the previous triangle. There are several ways of doing this construction and so this question is open ended and an opportunity to encourage experimentation. What we have done is to use the fact that the height is now $1/\sqrt{2}$ the height of the previous triangle. We started with a triangle of area 1 and reflected the mid-point of the hypotenuse of the first triangle in its base, thus arriving at the vertex of the second triangle. Do encourage your students to observe several congruent triangles, apart from the similar triangles. See Figure 4.

Conjecturing is an important mathematical skill and this is easily done in this task with students simply realizing from the figure that the table now reads to be read backwards, the areas are halved instead of being doubled. Students should be encouraged to record their conjectures, the objective being to realise that where the areas were increasing in the ratio 2^1, they are now decreasing in the ratio 2^{-1}.

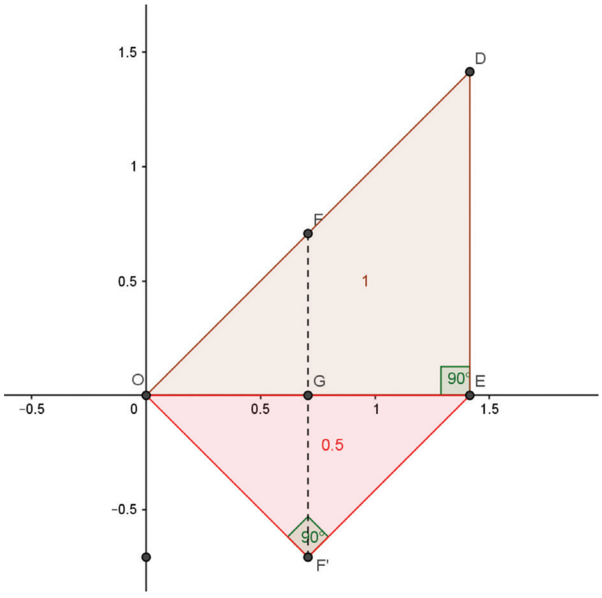

Figure 4

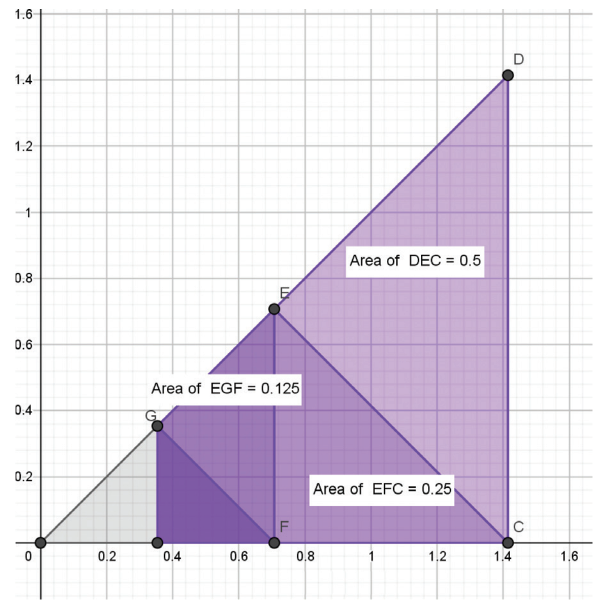

Figure 5

Task 5

Is it possible to tile the original triangle of area 1 with triangles whose areas are being halved? Will the original triangle be filled with these triangles or will the triangles 'spill out' as more and more are generated?

Teacher's Notes: A very preliminary, very visual approach to limits using a geometrical progression without mentioning either of these. This is intended for the more visual student who can use the ideas from the construction above to fill up the original triangle of area 1.

From the way the triangle fills up, it becomes clear that the construction of each new triangle is by finding the mid-point of the hypotenuse of the previous triangle and then making this half segment the base of the next triangle. Theoretically, it is possible to continue this process infinitely, without exiting from the first triangle. We would encourage teachers to write down this finding in the form of a mathematical statement; an example would be:

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots\ldots\ldots\ldots\ldots\ldots = 1$$

Verification is possible with the use of a calculator and this should be an exciting observation for students.

Task 6

Starting with a triangle of base 1 and height 2, generate a spiral of right angled similar triangles, building the base of each new triangle on the hypotenuse of the previous triangle.

What is the ratio of the sides and of the areas?

How can you generate a spiral of right angled similar triangles, starting with a triangle of base 1 such that the areas are in the ratio n (where n is a rational number)?

For what values of n does the spiral close?

Teacher's Note: The hypotenuse of the first triangle is $\sqrt{5}$, its area is 1 unit. For the next triangle to be similar, the sides have to increase in the ratio $\sqrt{5} : 1$ (as the hypotenuse is the new base). This makes the height $2\sqrt{5}$ units and the area will then be 5 square units.

Successive triangles will also have corresponding sides in the ratio $\sqrt{5} : 1$ with areas increasing in the ratio $5 : 1$.

If the ratio of the areas of the similar triangles is to be $n : 1$, then $n = \dfrac{A_2}{A_1} = \dfrac{\frac{1}{2}b_2h_2}{\frac{1}{2}b_1h_1} = \sqrt{n}\times\sqrt{n}$

The first triangle has base 1, hypotenuse = $\sqrt{n}$, then the height has to be $\sqrt{(n-1)}$.

Area will be $\frac{\sqrt{(n-1)}}{2}$

The next triangle has base $\sqrt{n}$, height $\sqrt{[n(n-1)]}$ and hypotenuse n, area will be $n\frac{\sqrt{(n-1)}}{2}$.

Clearly, we have to start by constructing the hypotenuse of the triangle with base 1, to be $\sqrt{n}$, and then generate succeeding triangles. Since $\sqrt{n}$ is constructible, we use the standard construction to generate the first triangle. See Figure 6, which is from the Std. 9 NCERT Text Book.

Task 7

Is it possible to construct new triangles in the clockwise direction for this new spiral?

What is the ratio of the sides in this case? What about the ratio of the areas?

Teacher's Note: Entirely possible, using similar triangles – the angle at the origin is reflected in the base of the first triangle and then the triangle is completed using the original base as the new hypotenuse. See Figure 7.

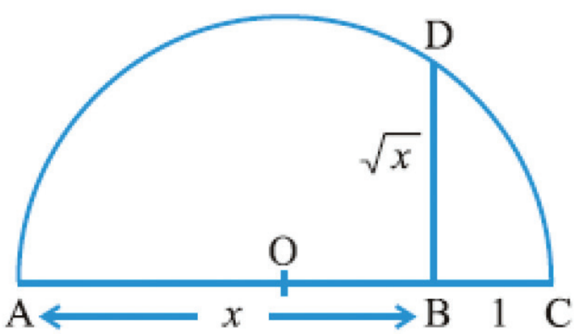

Figure 6

The sides will be in the ratio 1 : $1/\sqrt{n}$ and the areas in the ratio 1 : $1/n$. So while anti-clockwise rotations produce ratios of areas which are positive powers of n, the clockwise rotations produce ratios of areas which are negative powers of n.

Task 8

For what values of n does the spiral close?

Teacher's Note: Since the triangles are similar, the angles at the vertex are all equal for a given value of the ratio of areas (n : 1). The following table helps students to inspect the angle for values of n from 1 to 10. It becomes clear that the spiral closes for n = 2 and 4 for which the angles at the origin are 45° and 60° respectively. Both these are factors of 360°, the other factor in this range (remember that the angle is increasing (why?) but has to be acute) being 72°, but its cos value is not of the form $1/\sqrt{n}$.

Do encourage students to investigate beyond the first complete rotation for the spirals that close (n = 2 and n = 4). It is particularly interesting for them to note down the ratio of the sides and of the areas for the first triangle in each new loop!

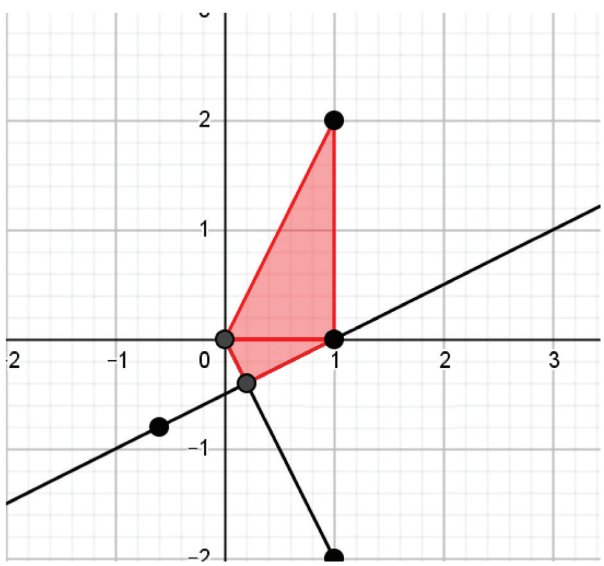

Figure 7

n [Ratio of Areas $n : 1$]	Base (b)	Opposite side (a)	Hypotenuse (c)	Cos θ = $\frac{b}{c}$	θ
2	1	1	$\sqrt{2}$	$\frac{1}{\sqrt{2}}$	45°
3	1	$\sqrt{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	54.7°
4	1	$\sqrt{3}$	2	$\frac{1}{2}$	60°
5	1	2	$\sqrt{5}$	$\frac{1}{\sqrt{5}}$	63.4°
6	1	…..	…..		
n	…...	……	…..		

Table 2: Angles at the origin

Conclusion

Isn't it amazing how going round and round the same question can make our investigation spiral? We hope that you will enjoy this investigation as much as we did creating it!

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