ClassRoom

Is a Parallelogram ever NOT A PARALLELOGRAM?

$\mathscr{C} \otimes \mathscr{M} \alpha \mathscr{C}$

A parallelogram may appear to be a very simple and basic shape of plane geometry, but its simplicity is deceptive; indeed, it possesses a lot of richness of structure. Much of this richness is revealed when we ask the following question: What characterises a parallelogram? In other words:

> What minimal properties must a quadrilateral have for us to know that it is actually a parallelogram?

The fact that a parallelogram can be defined in several different ways that turn out to be equivalent to each other is indicative of this richness. There is no other class of geometric objects which can be defined in so many different but equivalent ways.

The basic definition of a parallelogram is: A plane four-sided figure whose opposite pairs of sides are parallel to each other. That is, a plane four-sided figure $ABCD$ is a parallelogram if and only if $AB \parallel CD$ and $AD \parallel BC$ (see Figure 1).

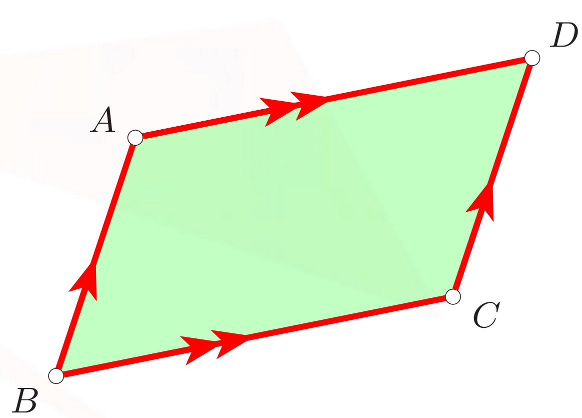

Figure 1

Keywords: Quadrilateral, parallelogram, SAS congruence, ASA congruence, SSS congruence

Here is a definition of a parallelogram which the reader may find unfamiliar, as it has been framed in the language of transformation geometry: A parallelogram is a quadrilateral with rotational symmetry of order 2. That is, if there exists a point O in the plane of a quadrilateral $ABCD$ such that a half-turn centred at O maps $ABCD$ back to itself, then $ABCD$ is a parallelogram.

Other alternative definitions

Here are some other ways in which a parallelogram can be defined. Each of these is equivalent to the basic definition given above. In each case, we have given a one-line indication of the proof. Throughout, we use '**iff**' as a short form for 'if and only if.' Also, throughout, 'quadrilateral' means 'planar quadrilateral.'

1. A four-sided figure $ABCD$ is a parallelogram iff $AB = CD$ and $AD = BC$. In words: A quadrilateral is a parallelogram iff both pairs of opposite sides have equal length. Proof: Use SSS congruence on an appropriate pair of triangles.
2. A four-sided figure $ABCD$ is a parallelogram iff $\measuredangle A = \measuredangle C$ and $\measuredangle B = \measuredangle D$. In words: A quadrilateral is a parallelogram iff both pairs of opposite angles have equal measure. Proof: Use ASA congruence on an appropriate pair of triangles.
3. A four-sided figure $ABCD$ is a parallelogram iff $AB \parallel CD$ and $AB = CD$. In words: A quadrilateral is a parallelogram iff for one pair of opposite sides, the sides are both parallel to each other and of equal length. Proof: Use SAS congruence on an appropriate pair of triangles.
4. A four-sided figure $ABCD$ is a parallelogram iff diagonals AC and BD bisect each other. In words: A quadrilateral is a parallelogram iff the diagonals bisect each other. Proof: Use SAS congruence on an appropriate pair of triangles.

These alternative definitions are well-known, so we shall not dwell on them further. Instead, we consider some fresh possibilities.

Do the following conditions characterise a parallelogram?

We offer below five different properties possessed by a parallelogram and ask in each case whether the property in question characterises a parallelogram; i.e., *if a planar quadrilateral possesses that property, is it necessarily a parallelogram?*

5. If $ABCD$ is a parallelogram, then each of its diagonals divides it into a pair of triangles with equal area. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that each of its diagonals divides it into two triangles that have equal area, is $ABCD$ necessarily a parallelogram?
6. If $ABCD$ is a parallelogram, then $AB = CD$ and $AD \parallel BC$. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that $AB = CD$ and $AD \parallel BC$, is $ABCD$ necessarily a parallelogram?
7. If $ABCD$ is a parallelogram, then $AB = CD$ and $\measuredangle A = \measuredangle C$. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that $AB = CD$ and $\measuredangle A = \measuredangle C$, is $ABCD$ necessarily a parallelogram?
8. If $ABCD$ is a parallelogram, then the sum of the squares of the sides equals the sum of the squares of the diagonals. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that

   $$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2,$$

   is $ABCD$ necessarily a parallelogram?
9. If $ABCD$ is a parallelogram, then the sum of the perpendicular distances from any interior point to the sides is independent of the location of the point. Does this condition characterise a parallelogram? In other words: If $ABCD$ is a quadrilateral such that the sum of the perpendicular distances from any interior point to the sides is independent of the location of the point, is $ABCD$ necessarily a parallelogram?

You may be intrigued to learn that three of these five conditions are genuine characterisations of a parallelogram; but two of them are not! We will leave it to you to find the two errant conditions, the ones that 'do not fit.' (Of course, in each of the cases, the assertion made in the first sentence is true. You may not be familiar with the last two assertions, items 8 and 9.)

For those who are impatient to know the answers, we study these five conditions in greater detail in the "How To Prove It" column, elsewhere in this issue.

References

[1] Jonathan Halabi, "Puzzle: proving a quadrilateral is a parallelogram" from *JD2718*, https://jd2718.org/2007/01/10/puzzle-proving-a-quadrilateral-is-a-parallelogram/

[2] Wikipedia, "Parallelogram" from https://en.wikipedia.org/wiki/Parallelogram

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