On an OPTIMIZATION PROBLEM

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Consider a solid sphere of radius r. Suppose it is melted and the molten mass is recast into two solid spheres of radii r_1 and r_2. Does the total surface area increase, decrease or remain the same, if it is assumed that there is no loss of matter in the whole process? Let us answer this question. Since the total volume remains the same, it must be that

$$\frac{4\pi}{3}r^3 = \frac{4\pi}{3}(r_1^3 + r_2^3), \tag{1}$$

and we are interested in knowing whether the ratio

$$\alpha = \frac{r_1^2 + r_2^2}{r^2} \tag{2}$$

is greater, less than or equal to one. Writing $x = r_1/r$ and $y = r_2/r$ we readily see that

$$x^3 + y^3 = 1. \tag{3}$$

Since $0 < x, y < 1$ it follows that $x^2 > x^3$ and $y^2 > y^3$, and these two inequalities combine to yield

$$\alpha = x^2 + y^2 > x^3 + y^3 = 1. \tag{4}$$

Thus the surface area increases. How large can α be? Can it be arbitrarily large, or is there a number it doesn't grow beyond? Since x and y are proper fractions, it is clear that α cannot be arbitrarily large. So, what is the upper bound? The algebraic identity

$$(x+y)\left(x^3+y^3\right) - \left(x^2+y^2\right)^2 = xy(x-y)^2 \tag{5}$$

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shows that $\alpha^2 \leq x+y < 2$. That is, $\alpha < \sqrt{2}$, so $\sqrt{2}$ is an upper bound for α. But note that this upper bound is not attained as x and y are strictly less than 1. Can we improve this upper bound? Observe that

$$(x+y)^2 \leq 2\left(x^2+y^2\right) = 2\alpha, \tag{6}$$

so

$$\alpha^2 \leq x+y \leq \sqrt{2\alpha} \tag{7}$$

and this leads to

$$\alpha \leq \sqrt[3]{2}. \tag{8}$$

Also, if $x = y = 1/\sqrt[3]{2}$, then $\alpha = \sqrt[3]{2}$. This shows that in order to maximize the total surface area of the two spheres derived from the original sphere, one needs to make them of equal size.

At this juncture, we naturally ask the following question:

> *If one wishes to maximise the total surface area of n spheres derived from a given sphere, then should they all be of the same radius?*

If $r_1, r_2, \ldots, r_n$ are the radii of the n spheres and we set $x_i = r_i/r$ for $i = 1, 2, \ldots, n$ and $s_n =$ the ratio of the sum of their surface areas to the surface area of the given sphere, then

$$s_n = x_1^2 + x_2^2 + \ldots + x_n^2 \tag{9}$$

and the problem reduces to maximizing s_n, subject to

$$x_1^3 + x_2^3 + \ldots + x_n^3 = 1. \tag{10}$$

Note that $\alpha = s_2$. This problem can be solved using multi-variable calculus, but there is a neat algebraic approach which uses the Cauchy-Schwarz inequality. For the benefit of the reader, we will state it and provide a sketch of proof.

Theorem (Cauchy-Schwarz Inequality)**.** *Let $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_n\}$ be any two sets of real numbers. Then*

$$\left|\sum_{j=1}^{n} a_j b_j\right| \leq \left(\sum_{j=1}^{n} a_j^2\right)^{1/2} \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}. \tag{11}$$

Equality holds if and only if there exists a constant λ such that $a_j = \lambda b_j$, for $1 \leq j \leq n$.

Proof. Write $A = \sum_{j=1}^{n} a_j^2$, $B = \sum_{j=1}^{n} b_j^2$, $C = \sum_{j=1}^{n} a_j b_j$. We need to show $C^2 \leq AB$.

If $B = 0$, then $b_j = 0$ for all j, so $C = 0$. Hence the inequality holds, trivially. We may therefore assume that $B \neq 0$. Since B is the sum of squares of real numbers, B must be positive.

Using the fact that the square of a real number is non-negative, we have

$$\begin{aligned} 0 &\leq \sum_{j=1}^{n} (Ba_j - Cb_j)^2 \\ &= B^2 \sum_{j=1}^{n} a_j^2 + C^2 \sum_{j=1}^{n} b_j^2 - 2BC \sum_{j=1}^{n} a_j b_j \\ &= B(BA - C^2). \end{aligned} \tag{12}$$

Since $B > 0$, it follows that $C^2 \leq AB$. We also infer that equality holds if and only if $Ba_j - Cb_j = 0$ for all j, $1 \leq j \leq n$. Taking $\lambda = C/B$, the condition for equality that $a_j = \lambda b_j$ for $1 \leq j \leq n$ is obtained. □

Application to the original problem. Returning to the original problem, we see that if we put $a_i = x_i^{1/2}$ and $b_i = x_i^{3/2}$ for $1 \le i \le n$, then by Cauchy-Schwarz we get

$$\begin{aligned} s_n^2 &= \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^2 \\ &\le \left(x_1 + x_2 + \cdots + x_n\right)\left(x_1^3 + x_2^3 + \cdots + x_n^3\right) \\ &= x_1 + x_2 + \cdots + x_n. \end{aligned}$$

Another application of Cauchy-Schwarz with $a_i = 1$ and $b_i = x_i$ for $1 \le i \le n$ yields

$$\left(x_1 + x_2 + \cdots + x_n\right)^2 \le (1 + 1 + \cdots + 1)\left(x_1^2 + x_2^2 + \cdots + x_n^2\right) = ns_n. \tag{13}$$

Combining the last two inequalities, we get

$$s_n^2 \le \sqrt{ns_n} \Rightarrow s_n \le \sqrt[3]{n}, \tag{14}$$

with equality if and only if $x_i = 1/\sqrt[3]{n}$ for $1 \le i \le n$.

Therefore, to maximize the total surface area, the spheres must be of radius $r/\sqrt[3]{n}$.

To get a feel of how fast the maximum value of the total surface area increases as n grows, consider the values of n from the set $\{10^{3k} : k = 1, 2, 3, \ldots\}$ and compute the maximum value of $s_{10^{3k}}$. It is 10^k. If $k = 1$, a ten-fold increase in the total surface area can be achieved by making 1000 spheres, each of radius $r/10$. The striking observation is that as n grows large, the volume of each sphere goes to zero, but the total surface area gets arbitrarily large. Perhaps in the limit the spheres are reduced to specks of dust and the entire mass is distributed as a two-dimensional surface.

References

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