FRACTIONAL TRIPLETS

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Consider this puzzle. *Three brothers – Youngest, Middle and Oldest – each receive some money in the form of some inheritance. Youngest keeps half the amount he receives and divides the balance equally among Middle and Oldest. After Middle receives his share, he too keeps half the amount and distributes the balance equally among Youngest and Oldest. Oldest in turn keeps half of the amount he now has (after receiving the shares from his brothers) and divides the remaining equally among Youngest and Middle. At the end, if all three of them have equal amounts, can we determine how much each of them had at the start? If so, how do we do so? If not, why not?*

This is an old puzzle with a slight modification. Let's explore the puzzle using numbers. We assume at the start that there is a bank from where we can draw as much money as we wish (natural numbers only; no fractions). We consider three players A, B and C. The order in which they play the game is A, then B, then C. Each player in turn takes a certain amount from the bank. For the players to act as required, A must take a multiple of 4. Let's say that A takes 4 units. How much should B take? Remember, B gets 1 from A, as A divides half of his amount equally among B and C. Obviously, B cannot take an even number, because an even number and 1 make an odd number. B also cannot take odd numbers like 1, 5, 9, 13, etc., because if we add 1 to these numbers, the resulting number will be an even number which is not divisible by 4. So B must take odd numbers like 3, 7, 11,… Suppose that B takes 7 units. B gets 1 from A, resulting in a total of 8. He keeps 4 for himself and divides the remaining amount equally between A and C. A now has 4 units, B has 4 units, and C has his own money plus 1 unit that he receives from A and 2 units that he receives from B. Now what quantity should C take so that he can complete the iterative steps? Like B, C too must have a quantity that is a multiple of 4 so that he can keep half for himself

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and divide the other half equally among A and B. So C's options are numbers like 1, 5, 9, 13,… Suppose that C takes 5 units; the result will be as shown below.

	A	B	C
	4 – 2	7 + 1	5 + 1
Balance	2 + 2	8 – 4	6 + 2
Balance	4 + 2	4 + 2	8 – 4
Balance	6	6	4

So if A, B and C start with 4, 7 and 5, they end up with 6, 6 and 4 respectively.

To understand any game better, it is a good idea to play it yourself. So take a pause and try playing the game with three players. Each player in turn chooses a certain number. At the end of every game, write down the quantities chosen at the start by the three players and the quantities that each has at the end of three rounds. For example, if A, B and C start with 4, 11 and 8, then this is what would happen.

	A	B	C
	4 – 2	11 + 1	8 + 1
Balance	2 + 3	12 – 6	9 + 3
Balance	5 + 3	6 + 3	12 – 6
Balance	8	9	6

At the end of the game, A is left with 8 units, B with 9, and C with 6.

Suppose the puzzle was, if the numbers left with A, B and C after the exchanges are 8, 9 and 6 respectively, how much did they have initially? How is this to be solved? Let's call this **Puzzle #2** and the puzzle at the beginning of this article **Puzzle #1**. We will return to this after some time. In the meantime, you can give some thought to how this can be solved.

After we play some games and observe the results, we find some relations and patterns. Table 1 shows the different results if A chooses 4 and B chooses 11.

Initial Amounts			Final Amounts		
A	B	C	A	B	C
4	11	4	7	8	4
4	11	8	8	9	6
4	11	12	9	10	8
4	11	16	10	11	10
4	11	20	11	12	12
4	11	24	12	13	14
4	11	28	13	14	16

Table 1

B could have taken 3, 7, 11, 15 or any number in this arithmetic progression. Table 1 shows the choices that C has, once A and B make a choice. We can see some patterns in the 'final amounts' table: *the initial amounts for A and B being the same, an increase of 4 units for C leads to increases in the final amounts of A, B and C by 1, 1 and 2, respectively.*

Let's fix rules for 'winning' the game. Say that the person who scores *least* in the end wins the game. Once A and B have chosen 4 and 11, C can choose 4, 8 or 12 to score the least. If he chooses 16, he ties with A (10 – 10). If he chooses 20, he ties with B. If he chooses any number beyond 24, he ends up with the maximum amount at the end of the game.

Let's try changing the rules for winning the game. The person who makes the maximum profit (final amount - initial amount) wins the game. Or the person who makes the maximum loss wins the game. Who gets an advantage in the game? Or is it a fair game?

A closer look at a few more game scores may help us see some more patterns and make more rules.

Initial Amounts			Final Amounts		
A	B	C	A	B	C
4	3	2	4	3	2
4	3	6	5	4	4
4	3	10	6	5	6
4	3	14	7	6	8

Table 2

Initial Amounts			Final Amounts		
A	B	C	A	B	C
8	6	0	7	5	2
8	6	4	8	6	4
8	6	8	9	7	6
8	6	12	10	8	8

Table 3

Initial Amounts			Final Amounts		
A	B	C	A	B	C
4	3	2	4	3	2
8	6	4	8	6	4
12	9	6	12	9	6
$4x$	$3x$	$2x$	$4x$	$3x$	$2x$
8	10	11	11	10	8
16	20	22	22	20	16
$8x$	$10x$	$11x$	$11x$	$10x$	$8x$
4	7	13	8	8	8
8	14	26	16	16	16
$4x$	$7x$	$13x$	$8x$	$8x$	$8x$

Table 4

The above tables show the patterns when different numbers are chosen as initial amounts. Let's generalise the game. When A chooses a number of the form $8a – 4$, B must choose a number of one of these forms: $16b + 3$, $16b + 7$, $16b + 11$, $16b + 15$. Accordingly, C must choose a number of one of these forms: $4c + 2$, $4c + 1$, $4c$, $4c + 3$. Similarly, when A chooses a number of the form $8a$, B must choose a number of one of these forms: $16b + 2$, $16b + 6$, $16b + 10$, $16b + 14$. Accordingly, C must choose a number of one of these forms: $4c + 1$, $4c$, $4c + 3$, $4c + 2$, respectively. Table #5 summarises this observation:

A	B	C
$8a – 4$	$16b + 3$	$4c + 2$
$8a – 4$	$16b + 7$	$4c + 1$
$8a – 4$	$16b + 11$	$4c$
$8a – 4$	$16b + 15$	$4c + 3$
$8a$	$16b + 2$	$4c + 1$
$8a$	$16b + 6$	$4c$
$8a$	$16b + 10$	$4c + 3$
$8a$	$16b + 14$	$4c + 2$

Table 5

Solving puzzles related to triplets

Let's go back to Puzzle #2, where it is given that after three exchanges, A, B and C are left with 8, 9 and 6, and we need to find the initial amounts. One can do it algebraically by assuming the initial amounts to be a, b and c and forming three equations. Solving the equations, we deduce the initial amounts. However, the answer can also be arrived at by studying patterns. Here's what one can do.

A must have started with a multiple of 4. In this example, A must have started with one of the following numbers: 4, 8, 12, 16, 20.

Case #1: Can A have started with 16 or 20? No, because after the first exchange itself, he would be left with 8 or more, and after the second and third exchanges, he would have still more at the end of the game. However, it is given that A has 8 left with him at the end of the game. So we rule out this possibility.

Case #2: Can A have started with 12? If so, then he will be left with 6 after the first exchange and would need only 2 more to reach 8 at the end of the game. For this to happen, B and C should give A 1 each, or one of them should give 0 and the other one should give 2. If B gives 1, then B should have 4. Then C will have 7 ($23 – 12 – 4 = 7$). With A's and B's share of 1 each, C will now have 9 which cannot be divided. Hence B cannot give 1 to A. Let's now consider B giving 2 to A. In that case, B should have 8. If A has 12 and B has 8, then C will have 3. Since C has 3, anything added to C will result in a minimum share of 1 going to A, which is not what we want. So we rule out this possibility as well.

Case #3: Can A have started with 8? If so, then B will have a number of one of these forms: $16x + 2$, $16x + 6$, $16x + 10$, $16x + 14$, and C will have a number of one of these forms: $4c + 1$, $4c$, $4c + 3$, $4c + 2$ (refer Table #5). Considering these points, B can have 2, 6, 10 or 14, which leaves C with the options (from 23 – A's amount – B's amount) 13, 9, 5 or 1, respectively.

- If B has 2, then after getting the share of 2 from A, B will have to give 1/4 of the share to A and 1/4 to C. Since C should have 13 (23 – A's amount – B's amount), after the second exchange, C will have 16 but he should have only 12 because he is left with 6 at the end. So this does not work out.

- If B has 6, then C will have 9 and after the second exchange, C will be left with 13 instead of 12.
- If B has 10, then C will have 5 and after the second exchange, C will be left with 10 instead of 12.
- We don't even have to check if B has 14 because after observing the earlier case, we are sure that C will be left with less than 10.

We conclude that A cannot have 8. Hence A has 4.

- **Case #4:** A has 4. According to Table #5, B will have 3, 7, 11 or 15 and accordingly C will have four different values. Again by taking four cases, we will be able to eliminate three of them and will be left with only one case which is the initial amount of 4, 11 and 8 respectively.

Using the general form, we can create puzzles where the final result after three exchanges is given and one has to work out the initial amounts that the three people had. There can be other strategies to tackle the puzzle which the reader will discover on his or her own.

Some points to ponder

Before we conclude, here are a few points to ponder.

1. The first few rows in Table #4 show that when we consider three numbers of the form $4x$, $3x$ and $2x$, the game will go into a loop. Let us call (4, 3, 2) a ***Repeating Fractional Triplet***.
2. A triplet of the form ($4x$, $7x$, $13x$) results in ($8x$, $8x$, $8x$) after three exchanges. Let's call (4, 7, 13) a ***Uniform Fractional Triplet***. Is there a ***Uniform Fractional Triplet*** not of the form ($4x$, $7x$, $13x$)? If yes, how many? If not, why not?
3. A triplet of the form ($8x$, $10x$, $11x$) results in ($11x$, $10x$, $8x$) after three exchanges. Let's call (8, 10, 11) a ***Reverse Fractional Triplet***. Is there a ***Reverse Fractional Triplet*** not of the form ($8x$, $10x$, $11x$)? If yes, how many? If not, why not?
4. The triplet (4, 15, 3) results in (8, 10, 4) after three exchanges. But it doesn't stop there. We can continue further and reach (7, 6, 9) after five exchanges. Is there a triplet for which more than five exchanges can be done? If so, what is the maximum number of exchanges that can be done with a given triplet? How many such triplets are there? Can we support it with a proof?
5. Instead of a triplet, let us consider a quadruplet (6, 5, 4, 3) where four persons play the game and four exchanges need to be done instead of three. After four iterations, (6, 5, 4, 3) results in (6, 5, 4, 3), i.e., the same quadruplet; so it is a ***Repeating Fractional Quadruplet***. Similarly, (8, 7, 6, 5, 4) is a ***Repeating Fractional Quintuplet***. Can we find more like these?
6. Can we find the general form for a ***Fractional Quadruplet*** and ***Fractional Quintuplet?***
7. Can we find new types of triplets other than ***Uniform, Repeating, Reverse?***

Can we solve Puzzle #1 when the final numbers (or the total of what remains with the three people) are not given? What do you think?

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