A CYCLIC KEPLER QUADRILATERAL & THE GOLDEN RATIO

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Introduction

A recent paper by Bizony (2017) discussed the interesting golden ratio properties of a Kepler triangle, defined as a right-angled triangle with its sides in geometric progression in the ratio $1 : \sqrt{\phi} : \phi$, where $\phi = \frac{1+\sqrt{5}}{2}$.

This inspired me to wonder what would happen if one similarly defined a 'Kepler quadrilateral' with sides in geometric progression with common ratio $\sqrt{\phi}$. (This is an example of the mathematical process of 'constructive defining,' whereby a new concept or object is defined by extension or modification of the definition of an already existing concept or object; compare De Villiers, 2017a & 2017b.) Would such a quadrilateral perhaps also exhibit some golden ratio, or other interesting properties?

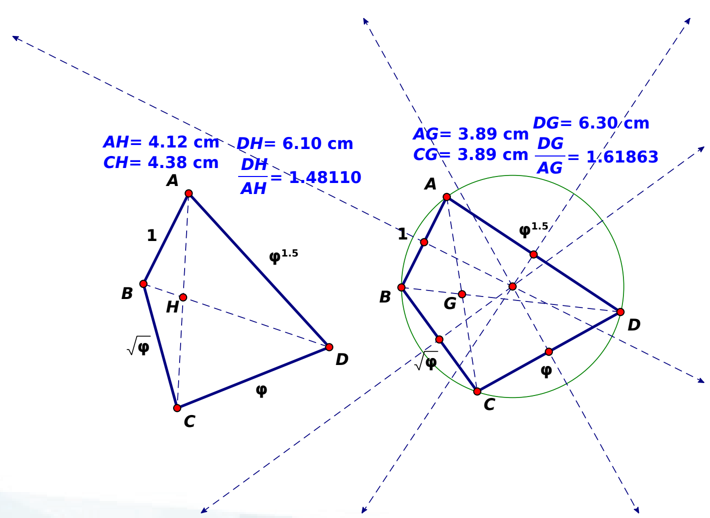

Figure 1

Keywords: Dynamic geometry, Kepler triangle, Kepler quadrilateral

A Conjecture about a Cyclic Kepler Quadrilateral

Proceeding to construct such a 'Kepler quadrilateral' $ABCD$ with sides in geometric progression as indicated by the first figure in Figure 1 produces a flexible quadrilateral with a changing shape. No interesting, invariant properties seemed immediately apparent. However, if $ABCD$ is dragged so that the perpendicular bisectors of the sides become concurrent (i.e., so that it becomes cyclic), as indicated by the second figure in Figure 1, it was observed as shown by measurements that not only did it seem that diagonal AC appeared to be bisected by diagonal BD, but also that $DG : AG = \phi$.

The reader is invited to explore these conjectures dynamically by visiting an interactive sketch at: http://dynamicmathematicslearning.com/cyclic-kepler-quadrilateral.html.

Though the dynamic geometry software is quite convincing about the truth of the conjectures, it doesn't *explain why* the result is true. For a satisfactory explanation, a deductive proof is required. Readers are now challenged to first try and prove the result themselves before reading further.

Proof

With reference to the second figure in Figure 1, we have from the sine area formula that

$$\text{Area } \Delta ABD = \frac{1}{2}\phi^{3/2}\sin A,$$
$$\text{Area } \Delta BCD = \frac{1}{2}\phi^{3/2}\sin C.$$

But since it is given that $ABCD$ is cyclic, angles A and C are supplementary, hence the sines of the two angles are equal. It follows that $area\Delta ABD = area\Delta BCD$. Therefore, diagonal BD bisects the area of the cyclic Kepler quadrilateral. It turns out that this is a necessary and sufficient condition for BD to bisect diagonal AC (compare Pillay & Pillay, 2010, pp. 16-17; Josefsson, 2017, p. 215), and, in addition, a proof is given as an Addendum at the end of this paper. Also note from the proof above that this bisecting diagonal property obviously generalizes to any cyclic quadrilateral with sides $AB : BC : CD : DA$ in geometric progression with common ratio r.

For the second property, again apply the sine area formula to obtain

$$\text{Area } \Delta ACD = \frac{1}{2}\cdot\phi\cdot\phi^{3/2}\sin D,$$
$$\text{Area } \Delta ABC = \frac{1}{2}\cdot 1\cdot\phi^{1/2}\sin D.$$

Hence,

$$\frac{\text{Area } \Delta ACD}{\text{Area } \Delta ABC} = \phi^2 = \frac{\text{perpendicular height from } D \text{ to } AC}{\text{perpendicular height from } B \text{ to } AC} = \frac{DG}{BG},$$

since the two right triangles with respective hypotenuses DG and BG respectively formed by the perpendiculars from D and B to AC are similar (two pairs of corresponding angles are equal). Therefore,

$$BG = \frac{DG}{\phi^2}. \tag{1}$$

From the intersecting chords theorem, $BG \times DG = AG^2$ and by substitution of (1) above into this equation, the desired result $\frac{DG}{AG} = \phi$ is obtained. The proof is now complete.

It is left to the reader to explore additional properties of a cyclic Kepler quadrilateral, which is called a ‘bisect-diagonal quadrilateral’ (because one diagonal is bisected by the other) by Josefsson (2017), who proves a long list of interesting results related to such quadrilaterals. For example, for the cyclic Kepler quadrilateral in Figure 1, diagonal BD can be determined from either one of the following equivalent formulae:

$$\sqrt{\frac{1+\phi+\phi^2+\phi^3}{2}}=\sqrt{\frac{\phi^4-1}{2(\phi-1)}}=\sqrt{\frac{(\phi^2+1)(\phi+1)}{2}}.$$

Concluding Remark

This little exploration would be suitable for high school level students knowing some trigonometry and circle geometry. It therefore provides an accessible, non-routine challenge as well as illustrating how new results can sometimes be discovered experimentally by the ‘what-if’ extension of older known results, and using dynamic geometry as an effective investigative tool.

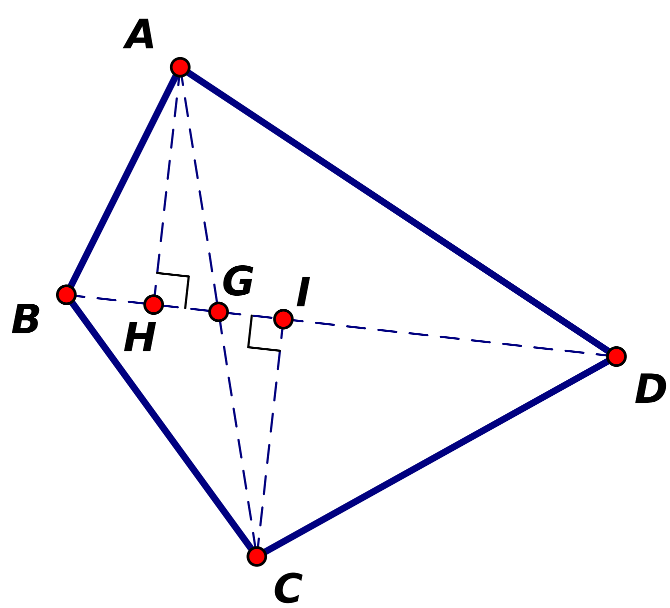

Figure 2

Addendum: Lemma

The area of a quadrilateral $ABCD$ is bisected by diagonal BD if and only if BD bisects AC.

Proof. Consider Figure 2. If area ABD = area BCD, it follows that the perpendicular heights AH and CI to the common base BD have to be equal. Hence, triangles AHG and CIG are congruent and therefore, $AG = CG$. The converse follows similarly and is left to the reader. (Note: the result is also true for a concave quadrilateral $ABCD$).

References

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