Problem Corner

ADVENTURES IN PROBLEM SOLVING

Miscellaneous Problems from the RMOs

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In this edition of 'Adventures' we study a few miscellaneous problems, some from past RMOs.

As usual, we pose the problems first and give the solutions later in the article, thereby giving you an opportunity to work on the problems.

Problems

(1) Show that the equation

$$a^2+(a+1)^2+(a+2)^2+(a+3)^2+(a+4)^2 + (a+5)^2+(a+6)^2=b^4+(b+1)^4$$

has no solutions in integers a and b. (RMO 2017, #2)

(2) Given that x is a non-zero real number such that x^4+1/x^4 and x^5+1/x^5 are rational numbers, prove that $x+1/x$ is a rational number. (RMO 2013, Paper 4, #4)

(3) Let ABC be an isosceles triangle in which $\measuredangle A=100^\circ$ and $\measuredangle B=40^\circ=\measuredangle C$. Let side AB be extended to a point D such that $AD=BC$. Find $\measuredangle BCD$.

Solutions

(1) *Show that the equation*

$$a^2+(a+1)^2+(a+2)^2+(a+3)^2+(a+4)^2 + (a+5)^2+(a+6)^2=b^4+(b+1)^4$$

has no solutions in integers a and b.

Keywords: Rational number, isosceles, triangle, exterior angle, trigonometric identity, modulus

Solution. The fact that on the left side of this equation we have the sum of the squares of **seven** consecutive integers is a strong clue to how we must proceed. Namely, we can examine both sides of the equation modulo 7. (This strategy offers us a good chance of succeeding. Of course, nothing can be guaranteed....) On the left side, the coefficient of a^2 is 7, which is $0 \pmod 7$. The coefficient of a is $2(1+2+3+4+5+6)$ which also is $0 \pmod 7$. Finally, the constant term is

$$1^2+2^2+3^2+4^2+5^2+6^2=\frac{6\times 7\times 13}{6},$$

and this yet again is $0 \pmod 7$. So the quantity on the left is identically $0 \pmod 7$.

Now we must check what happens on the right side of the equation, where we see the sum of the fourth powers of two consecutive integers. Let us first list the residues left when we divide the sequence of fourth powers (0, 1, 16, 81, 256, ...) by 7; we get:
$0,\ 1,\ 2,\ 4,\ 4,\ 2,\ 1,\ 0,\ 1,\ 2,\ 4,\ 4,\ 2,\ 1,\ \ldots$. We see a repeating sequence of residues:

$$\underbrace{0,\ 1,\ 2,\ 4,\ 4,\ 2,\ 1}, \ \underbrace{0,\ 1,\ 2,\ 4,\ 4,\ 2,\ 1},\ \ldots .$$

The string $0, 1, 2, 4, 4, 2, 1$ of length 7 repeats indefinitely. To verify that it does repeat, we only need to verify that $(n+7)^4-n^4$ is a multiple of 7 for all integers n. But this verification is routine.

Finally, it is easy to verify that no two consecutive members of the string 0, 1, 2, 4, 4, 2, 1 yield a sum which is $0 \pmod 7$. It follows that the expression $b^4+(b+1)^4$ is not a multiple of 7 for any integer b. Hence the given equation has no integer solutions.

Remark. In the same way, we can show that the following equations have no solutions in integers a and b:

$$a^2+(a+1)^2+\cdots+(a+5)^2+(a+6)^2=b^2+(b+1)^2,$$

$$a^2+(a+1)^2+\cdots+(a+9)^2+(a+10)^2=b^2+(b+1)^2,$$

$$a^2+(a+1)^2+\cdots+(a+9)^2+(a+10)^2=b^2+(b+1)^2+(b+2)^2+(b+3)^2.$$

(2) *Given that x is a non-zero real number such that x^4+1/x^4 and x^5+1/x^5 are rational numbers, prove that $x+1/x$ is a rational number.*

Solution. Write S_n to denote the quantity x^n+1/x^n; then $S_0=2$ and $S_{-n}=S_n$. Note the following identity:

$$\left(x^m+\frac{1}{x^m}\right)\cdot\left(x^n+\frac{1}{x^n}\right)=x^{m+n}+\frac{1}{x^{m+n}}+x^{m-n}+\frac{1}{x^{m-n}}.$$

From this we see that

$$S_m\cdot S_n=S_{m+n}+S_{m-n},$$

$$S_{2m}=S_m^2-2.$$

From these relationships, we see that if any three of the four quantities S_m, S_n, S_{m+n}, S_{m-n} are rational numbers, then so is the fourth one. Also, if S_m is rational, then so is S_{2m}. Now observe the following:

- Since S_4 and S_5 are rational (given), so are S_8 and S_{10}.

- Since $S_4S_2 = S_6 + S_2$ and $S_8S_2 = S_{10} + S_6$, it follows that

$$\begin{aligned} S_{10} &= S_8S_2 - S_6 \\ &= S_8S_2 - S_4S_2 + S_2 \\ &= S_2\left(S_8 - S_4 + 1\right). \end{aligned}$$

  Since S_4, S_8 and S_{10} are rational, so is S_2.

- Since $S_4S_2 = S_6 + S_2$ and S_2, S_4 are rational, so is S_6.
- Finally, since $S_5S_1 = S_6 + S_4$ and S_4, S_5, S_6 are rational, so is S_1.

It would be of interest to explore the following question:

Let x be a nonzero real number, and let $S_n = x^n + 1/x^n$. Suppose that S_m and S_{m+1} are rational numbers, for some positive integer $m > 1$. Does it necessarily follow that S_1 is rational?

(3) *Let ABC be an isosceles triangle in which $\measuredangle A = 100°$ and $\measuredangle B = 40° = \measuredangle C$. Let side AB be extended to a point D such that $AD = BC$. Find $\measuredangle BCD$.*

Solution. We offer three solutions; the first one uses trigonometry, while the other two use only ideas from pure geometry.

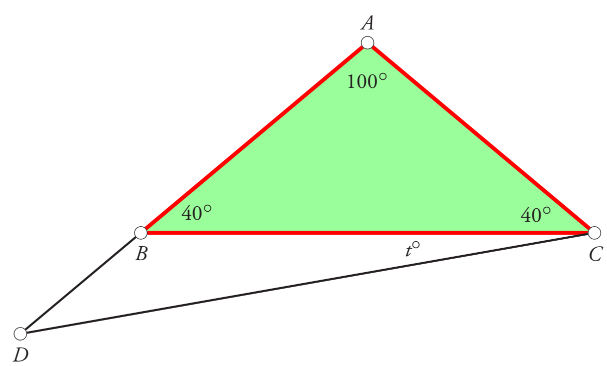

Figure 1

Trigonometric solution. Let $t°$ denote the measure of $\measuredangle BCD$. Then we have, via the sine rule:

$$\text{From } \triangle ADC: \quad \frac{AD}{AC} = \frac{\sin(40+t)°}{\sin(40-t)°},$$

$$\text{From } \triangle ABC: \quad \frac{BC}{AC} = \frac{\sin 100°}{\sin 40°}.$$

Since $AD = BC$ (given), we obtain:

$$\frac{\sin(40+t)°}{\sin(40-t)°} = \frac{\sin 100°}{\sin 40°}.$$

Since $\sin 100° = \sin 80° = 2 \cdot \sin 40° \cdot \cos 40°$, this yields:

$$\frac{\sin(40+t)°}{\sin(40-t)°} = 2\,\cos 40°.$$

This equation needs to be solved for t. The solution can be effected in many ways; I'm sure that readers will find some ways of their own. Here is one that occurred to me. Using the fact that $\sin 30^\circ = 1/2$ and $\cos 40^\circ = \sin 50^\circ$, we rewrite the above relation as:

$$\sin(40+t)^\circ \cdot \sin 30^\circ = \sin(40-t)^\circ \cdot \sin 50^\circ.$$

Next, using the identity $2 \cdot \sin x \cdot \sin y = \cos(x-y) - \cos(x+y)$, we get:

$$\cos(10+t)^\circ - \cos(70+t)^\circ = \cos(10+t)^\circ - \cos(90-t)^\circ.$$

Hence:

$$\cos(70+t)^\circ = \cos(90-t)^\circ.$$

Since $t^\circ < 40^\circ$ ("exterior angle of a triangle strictly exceeds each of the interior opposite angles"), both $70+t$ and $90-t$ lie in the interval $[0, 180]$; and since the cos function does not repeat any values in the interval $[0^\circ, 180^\circ]$, the above relation yields $70+t = 90-t$. Hence $t = 10$, i.e., $\measuredangle BCD = 10^\circ$.

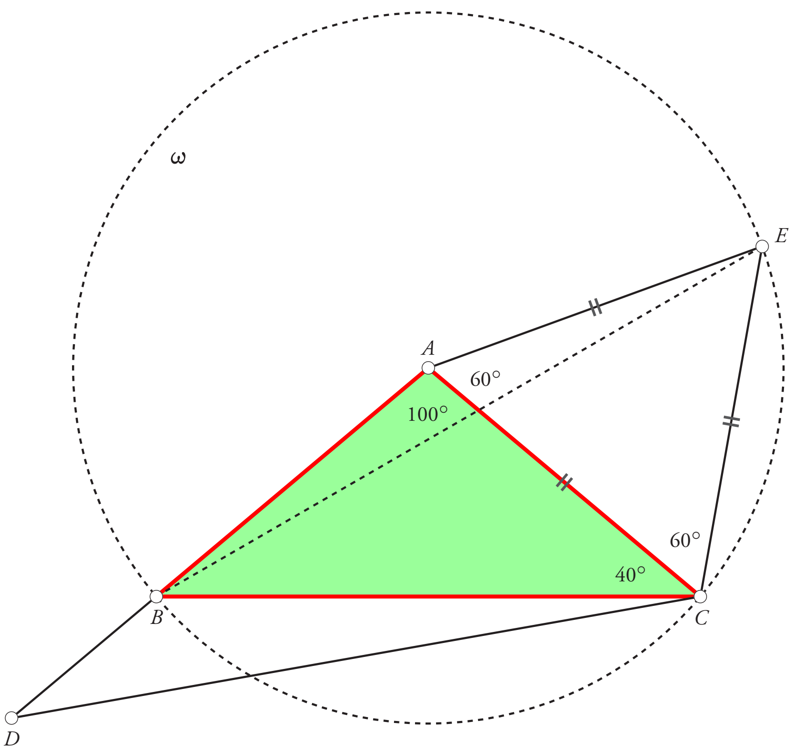

Figure 2

Geometric solution–I. Draw a circle ω centred at A and passing through C. Draw equilateral triangle ACE with AC as a side, lying outside $\triangle ABC$. (See Figure 2.) Naturally, E lies on circle ω. Join BE.

Then $\triangle BCE \cong \triangle DAC$ (side-angle-side or 'SAS' congruence: $BC = DA$, $CE = AC$ and $\measuredangle BCE = \measuredangle DAC$), hence $\measuredangle ACD = \measuredangle CEB = 100^\circ \div 2 = 50^\circ$, hence $\measuredangle BCD = 10^\circ$.

(Solution sent in by Dipak Bidap of Bangalore.)

Geometric solution–II. Draw the angle bisector CP of $\measuredangle ACB$, with P on side AB. Locate Q on side BC such that $CQ = CP$. Also locate R on side BC such that $\measuredangle PRC = 100^\circ$. (See Figure 3.)

We have now: $\triangle APC \cong \triangle RPC$, hence $AP = RP$.

Next, $\measuredangle PQB = 100^\circ$, so $\measuredangle PQR = 80^\circ = \measuredangle PRQ$, hence $PQ = PR$, hence also $AP = PQ$.

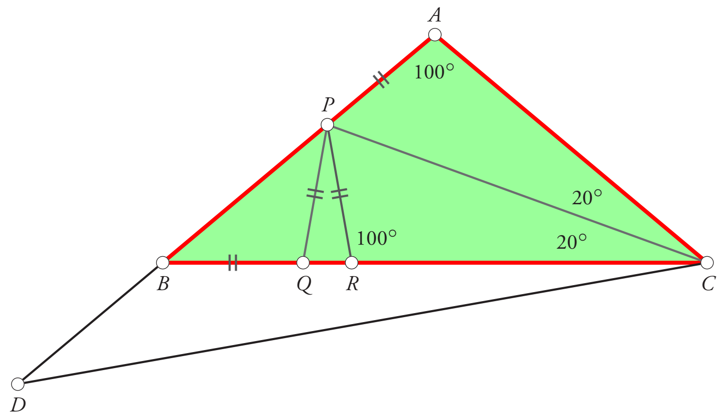

Figure 3

Again, $\triangle QPB$ has angles 100°, 40°, 40°, so $QP = QB$, i.e., $BQ = AP$. Since $BC = AD$, we get by subtraction $CQ = PD$. Hence $PD = PC$.

Since $\measuredangle DPC = 120^\circ$, it follows that $\measuredangle PCD = (180^\circ - 120^\circ)/2 = 30^\circ$. Hence $\measuredangle BCD = 10^\circ$.

(Solution sent in by K V Bhagwat of Mumbai.)

Remark. Note the streamlined elegance of the pure geometry solutions, in comparison with the trigonometric solution. This is a typical scenario: solutions based on pure geometry always look extremely elegant when placed alongside solutions based on trigonometry or coordinate geometry. But this elegance is highly deceptive; finding such solutions is no easy matter.

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