Problems for the SENIOR SCHOOL

Problem Editors: PRITHWIJIT DE & SHAILESH SHIRALI

Problem VII-1-S.1
Two hundred students are positioned in 10 rows, each containing 20 students. From each of the 20 columns thus formed, the shortest student is selected, and the tallest of these 20 (short) students is labelled A. These students return to their initial places. Next, the tallest student in each row is selected, and from these 10 (tall) students, the shortest is labelled B. Who is taller, A or B?

Problem VII-1-S.2
Given 13 coins, each weighing an integral number of grams. It is known that if any coin is removed, then the remaining 12 coins can be divided into two groups of 6 with equal total weight. Prove that all the coins are of the same weight.

Problem VII-1-S.3
Show that there are infinitely many positive integers A such that $2A$ is a square, $3A$ is a cube and $5A$ is a fifth power.

Problem VII-1-S.4
An infinite sequence of positive integers $a_1, a_2, \ldots, a_n, \ldots$ satisfies the condition $\sum_{k=1}^{m} a_k^3 = \left(\sum_{k=1}^{m} a_k\right)^2$, i.e.,

$$a_1^3 + a_2^3 + a_3^3 + \cdots + a_m^3 = (a_1 + a_2 + a_3 + \cdots + a_m)^2$$

for each positive integer m. Determine the sequence.

Problem VII-1-S.5
The function $f(n) = an + b$, where a and b are integers, is such that for every integer n, the numbers $f(3n+1)$, $f(3n)+1$ and $3f(n)+1$ are three consecutive integers in some order. Determine all such functions $f(n)$.

Keywords: Integer, coin, square, cube, infinite, sequence, function, inversion, round-robin tournament

Solutions of Problems in Issue-VI-3 (November 2017)

Solution to problem VI-3-S.1

The numbers $1, 1/2, 1/3, \ldots, 1/2017$ *are written on a blackboard. A student chooses any two numbers from the blackboard, say x and y, erases them and instead writes the number* $x + y + xy$*. She continues to do so until there is just one number left on the board. What are the possible values of the final number?*

Observe that

$$xy + x + y = (x+1)(y+1) - 1,$$

and if

$$x = (u+1)(v+1) - 1, \quad y = (p+1)(q+1) - 1,$$

then

$$xy + x + y = (u+1)(v+1)(p+1)(q+1) - 1.$$

Thus at the end of the process the number on the board is

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{2018}{2017} - 1 = 2018 - 1 = 2017.$$

Solution to problem VI-3-S.2

The numbers $1, 2, 3, \ldots, n$ *are arranged in a certain order. One can swap any two adjacent numbers. Prove that after performing an* ***odd*** *number of such operations, the arrangement of the numbers thus obtained will differ from the original one.*

Let $a_1, a_2, a_3, \ldots, a_n$ be a random permutation of the numbers $1, 2, 3, \ldots, n$. We say that the numbers a_i and a_j give rise to an *inversion* if $i < j$ but $a_i > a_j$. After every swap, the number of inversions either increases or decreases by 1. Thus the parity of the number of inversions in the arrangement is changed. Therefore, after an odd number of operations, the parity of the number of inversions in the final arrangement will be different from the parity of the number of inversions in the initial arrangement, hence the arrangement must differ from the original one.

Solution to problem VI-3-S.3

At each of the eight corners of a cube, write $+1$ *or* -1*. Then, on each of the six faces of the cube, write the product of the numbers at the four corners of that face. Add all the fourteen numbers so written down. Is it possible to arrange the numbers* $+1$ *and* -1 *at the corners initially in such a way that this final sum is zero?*

Let $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ and x_8 be the numbers written at the corners. Then, the final sum is given by

$$\sum_{i=1}^{8} x_i + x_1x_2x_3x_4 + x_5x_6x_7x_8 + x_1x_4x_5x_8 + x_2x_3x_6x_7 + x_1x_2x_5x_6 + x_3x_4x_7x_8.$$

As there are fourteen terms in the above sum and each term is $+1$ or -1, the sum will be zero only if some seven terms are $+1$ each and the remaining seven terms are -1 each. But, the product of the fourteen terms is

$$(x_1x_2x_3x_4x_5x_6x_7x_8)^4 = (\pm 1)^4 = +1.$$

Therefore, it is not possible to have an odd number of -1's in the above sum. We conclude that the desired arrangement is not possible.

Solution to problem VI-3-S.4

At a party, it is observed that each person knows 20 *others. Also, for each pair of persons who know one another, there is exactly one other person whom they both know. Further, for each pair of persons who do not know one another, there are exactly* 6 *other persons whom they both know. Also, if A and B are present in the party and A knows B, then B knows A. Determine, with proof, the number of people at the party.*

Pick a person u. We count in two ways all pairs of persons (v, w) such that u and v are distinct, v knows w, v knows u, and u does not know w. First, there are 20 persons v that know u, and, for each such v, there are 19 persons (excluding u) who know v and of these, exactly one knows u, so there are 18 persons w that know v but do not know u. So the number of pairs (v, w) as above is $20 \times 18 = 360$.

On the other hand, if n is the total number of people at the party, there are $n - 21$ people $w(\neq u)$ that do not know u, and for each such w, there are 6 persons v that know both u and w. So the total number of pairs (v, w) as above is $6(n - 21)$. Hence $6(n - 21) = 360$, and so $n = 81$.

Solution to problem VI-3-S.5

Suppose there are k teams playing a round-robin tournament; that is, each team plays against every other team. Assume that no game ends in a draw. Suppose that the i-th team loses l_i games and wins w_i games. Show that $\sum_{i=1}^{k} l_i^2 = \sum_{i=1}^{k} w_i^2$.

Since there are no draws, we must have

$$\sum_{i=1}^{k} l_i = \sum_{i=1}^{k} w_i.$$

Also, $\sum_{i=1}^{k} (l_i + w_i) = k - 1$. Therefore

$$\begin{aligned}
\sum_{i=1}^{k} l_i^2 - \sum_{i=1}^{k} w_i^2 &= \sum_{i=1}^{k} \left(l_i^2 - w_i^2\right) \\
&= \sum_{i=1}^{k} \left(l_i - w_i\right)\left(l_i + w_i\right) \\
&= (k-1) \sum_{i=1}^{k} \left(l_i - w_i\right) = 0.
\end{aligned}$$

The desired conclusion follows.