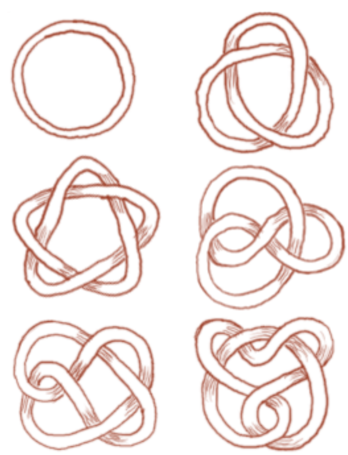
Tying it up …

KNOT THEORY

… without loose ends

RAMYA RAMALINGAM

Knot theory is an important sub-field of **topology** that studies the properties of different kinds of knots. This subject is a fairly new and still developing branch of mathematics. Interestingly, the roots of this subject originate in physics, not math. Physicists at one time believed that atoms and molecules were configurations of knotted thread. Although later models of the atom abandoned this postulate, knot theory came into its own in mathematics and in other branches of science.



Knot theory has applications in other mathematical branches as well – it is used substantially in graph theory, which in turn has implications in computer science while studying networks, data organization and computational flow. Knot theory also has uses in biology – it turns out that in certain organisms, DNA often twists itself up into knots which results in a host of different properties and sometimes problems for the organism. Knowledge of the properties of knots can be indispensable in studying this.

Keywords: Topology, knot, string, projection, tri-colourable, Reidemeister, unknot, trefoil, twist, poke, slide

Features

What is Topology?

In Euclidean geometry we have the notion of congruence of triangles. One way of approaching this topic is through the study of ***functions*** or ***mappings*** or ***transformations*** from the plane into itself. To start with, let us consider only functions which have the property that *the distance between any pair of points remains unaltered as a result of the mapping*. Such a mapping is known as an ***isometry*** ('iso' = same, 'metre' = distance). Under such mapping, any figure is mapped to a figure which is congruent to itself. Here the word 'congruence' is used in its usual sense. But we can turn the definition around and instead define congruence in terms of the mapping. That is, if a figure A can be mapped to a figure B using such a mapping, we say that B is congruent to A, and the study of all properties of these figures which remain unchanged as a result of these mappings is what we call ***Euclidean geometry***.

Note that the class of mappings allowed is of critical importance. If we enlarge the class, the notion of congruence changes accordingly.

The class of isometric mappings is a highly restricted one; so let us replace it with something larger. If we consider instead, functions which have the following property: *for any three points A, B, C which are mapped to the points A', B', C' respectively, angle ABC must be equal to angle A'B'C'*, then we get what is ordinarily called ***similarity geometry***.

In Topology, the functions permitted belong to a much bigger class. They are what are known as ***Homeomorphisms***. Rather than give a technical definition, let us say simply that they refer to ***continuous deformations***. Thus, we allow stretching, contracting, twisting, shearing and so on; but we do not allow ruptures or tears. If an object A in 3-D space can be transformed into another object B using continuous deformations, and B in turn can be transformed into A using continuous deformations, we say that A and B are ***topologically indistinguishable***.

Using functions from within this class, it is easy to see that:

- A circle of any size is topologically indistinguishable from an ellipse of any size and any eccentricity.
- A circle of any size is topologically indistinguishable from a rectangle of any size and any shape.
- A line segment of any length is topologically indistinguishable from a planar arc of any shape and any length, provided the arc does not intersect itself at any point.
- A doughnut is topologically indistinguishable from a coffee cup (assuming that the cup is of the usual kind, with a single handle!), and both these shapes are topologically distinct from a saucer.

The study of which shapes are topologically indistinguishable from which other shapes is an informal way of understanding the term ***Topology***.

Organic chemistry can also use knot theory in differentiating mirror image molecules (which can have astoundingly different properties) from each other.

What is a Knot?

We start with the most basic question about this topic: ***What exactly is a knot?*** In simple terms, a knot can be thought of as a piece of string crossed in a certain manner, the ends being tied together. The tying of ends is particularly important to the definition as any pattern of an open-ended string can be manipulated into any other pattern through continuous deformation. Thus, without closing the ends, it becomes impossible to differentiate between knots, an essential requirement in their study.

Some Basic Knots

1. **The Unknot**: This is also known as the trivial knot and is the simplest knot of all. It is simply a loop; the simplest way to look at it is this way – connect two ends of a string together without knotting the string or crossing it over itself at any point.

Figure 1

2. **The Trefoil**: The next simplest knot surprisingly shows up in a number of places in nature and microscopic organisms. Its simplest picture has 3 crossings. It is also called the overhead knot, and is the closed-end version of the simplest knot most people use to tie string.

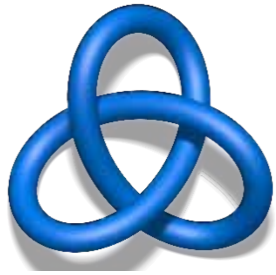


Figure 2

3. **Figure-8 knot**: The simplest viewing of this knot has 4 places where the string crosses itself. It is called so because one of its most common projections looks like the digit eight.

Figure 3

Experiment 1: Take a long piece of string; it should be sturdy but deformable. First, create the unknot by connecting the two ends of the string together. Next, deform it by passing the string over itself and creating different crossings and loops – but do not untie the string or cut it at any place. After you have made a number of crossings, place the knot on a table. Simply by looking at it, can you tell that it is the unknot, or does it look different?

Experiment 2: Now take another piece of string and create the trefoil knot by using Figure 2. Does it look like it could be the unknot? Try moving the knot around and deforming it. Is it possible to make the trefoil look like the simple version of the unknot? You could try to make the Figure-8 knot as well and see if you can transform it into either the trefoil or the unknot.

Equivalent Knots

Two knots are considered to be the same, or equivalent to one another, if you can 'deform' one into the other without breaking the knots open. While this is not a formal definition, it is intuitively understandable. If you imagine yourself to be holding the knot, and can somehow twist it or turn it to look like another knot, clearly the two must be the same. But there is still the need for a more official (rigorous?) definition, which leads to the questions:

1. How do we prove that two knots are the same?
2. How do we prove that two knots are different?

At first glance the two problems seem to be the same, but the second question is significantly more difficult to answer. We will try to provide a few things that might help.

Definitions

A ***projection*** is a 2-dimensional picture of a 3-dimensional knot.

The places where two strands of a projection meet is called a ***crossing***. The strand that goes over is called an *over-crossing* and the strand that goes under is called an *under-crossing*.

A ***strand*** of a projection is a piece of the knot cut off on both ends by a crossing. Basically, while drawing a knot projection, each strand corresponds to the longest piece of the knot you can draw at that point without lifting your pencil off the paper.

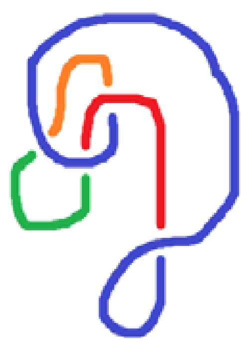


Figure 4

All of the differently coloured pieces of the knot represent strands.

A ***link*** is a collection (or a union) of multiple knots, possibly linked or knotted together. An *n-component link* is a link that consists of n individual knots.

A ***planar isotopy*** of a projection is a manipulation of a part of the knot in 2D space (by shrinking, straightening, enlarging) that does *not* change the number of crossings of the projection.

An ***ambient isotopy*** of a projection is a manipulation of the knot in 3D space that can change crossings. The only restriction is that the knot may not be cut anywhere.

Note: A knot can have a large number of differing projections.

The Reidemeister Theorem

The Reidemeister Theorem (named after the German mathematician Kurt Reidemeister, 1893 – 1971) states that *two projections are of the same knot if and only if either of the projections can be transformed into the other using a series of planar isotopies and/or Reidemeister moves*. We now explain what this means, but we shall not try to prove the theorem here.

We first need to explain the term ***'Reidemeister move'***. It turns out that there are three types of moves that encompass the ways you can manipulate a knot, provided you do not cut it anywhere. The three moves are called:

i. **The Twist (RM 1)**: If you have a straight piece of the knot, and twist it once so as to create a single crossing, you have performed Reidemeister Move 1; see Figure 5.

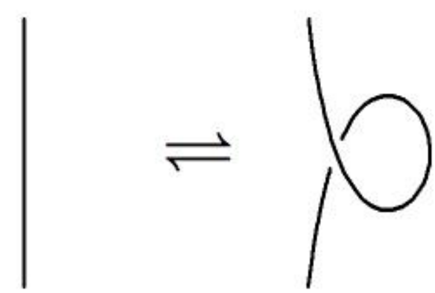


Figure 5

ii. **The Poke (RM 2)**: To perform Reidemeister Move 2, you need to push one part of the knot under (or over) another part of the knot so you get two under-crossings (or over-crossings) beside each other; see Figure 6.

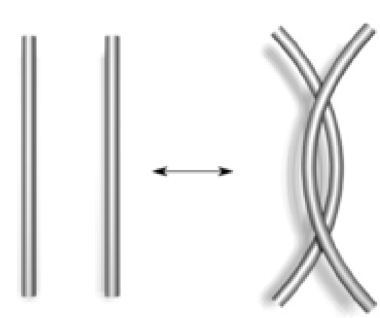


Figure 6

iii. **The Slide (RM 3)**: If a part of the knot goes under (or over) two other pieces which cross each other, that part can be slid under (or over) the crossing to the other side – this is Reidemeister Move 3; see Figure 7.

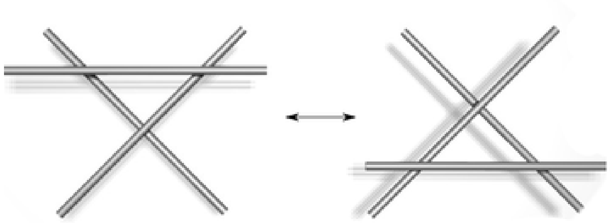
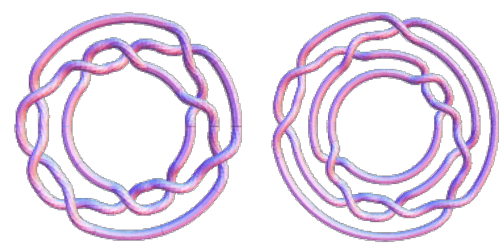


Figure 7

If two knot diagrams are projections of the same knot, the Reidemeister theorem clearly makes it easier to establish this fact. If we can use the Reidemeister moves to make one projection resemble the other, this theorem tells us they are the same knot.

But if we find ourselves unable to do this, we cannot assume that the two knot projections are different; perhaps we just made the wrong set of Reidemeister moves! In order to actually differentiate between knots, we need an *invariant*; something that does not change for a knot, no matter what projection of the knot is used. Then, if two different projections have different invariants, we can show that the two knots are different. For this purpose, something called the *tri-colourability* of a knot was defined.



What is an invariant?

Invariance is a powerful tool in mathematics. It is often used when we are trying to prove that something is not possible – which can sometimes be far more difficult to do than to prove that something is possible. The key step here is to find some quantity which does not change as we apply various transformations to the configuration. Many of the famous impossibility results in mathematics are proved via identification and skillful use of a suitable invariant. For example, it may be that there is a certain well-defined quantity whose parity does not change as a result of the permitted transformations, but such that the parities of the quantities associated with two given states are different. In such a case, it should be obvious that there can be no sequence of transformations which will take you from one state to the other state. A well-known problem of this genre is that of finding a method using Euclidean geometry tools (i.e., compass and unmarked straight edge) to trisect an arbitrary angle. The ancient Greek geometers struggled with this problem but did not make any headway. A full two millennia later, in the first half of the nineteenth century, it was shown that no such procedure can exist. This is an example of an impossibility result. In our context we may use invariance to help us with the second question – namely, showing that two knots are not the same.

Invariants are not used only for proving impossibility results. Many beautiful results relating to triangles and the conic sections can be proved using notions of invariance. Likewise, certain results in elementary number theory relating to divisibility can be proved using such notions.

Tri-colourability

A knot is defined as ***tri-colourable*** if one can colour the strands of the projection, using only 2 or 3 colours, such that at each crossing, either all strands are the same colour or all strands are of different colours.

Tri-colourability is useful as it turns out to be an invariant for any given knot – that is, if one projection of a knot is tri-colourable, then all projections of that knot are tri-colourable.

Proof

To prove that tri-colourability does not change with the projection of a knot, all we need to do is show that whether or not a projection is tri-colourable is not affected by any Reidemeister moves. This should be evident because any projection of a knot can be transformed into any other projection of the same knot using only Reidemeister moves.

RM-1

When we go from an untwisted strand to a twist (in which we can see two strands), it is clear from the diagram that the twisted projection can be coloured in such a way that preserves tri-colourability. When we look at the reverse direction (twisted to untwisted), we can see there is only one way to colour Strand 1 and Strand 2 such that the diagram is tri-colourable; that is, both strands have to be the same colour. In this case as well, we can see tri-colourability is preserved by the untwist. Thus, tri-colourability is preserved by Reidemeister Move 1.

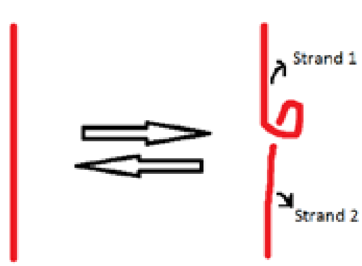


Figure 8

RM-2

In Reidemeister Move 2, there are two possibilities: the two strands, one of which 'pokes' over the other, may be of the same colour (Figure 9), or they may be of two different colours (Figure 10). In both cases, we can see that tri-colourability can be preserved.

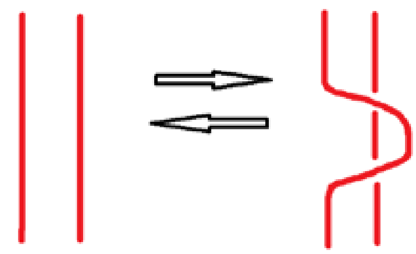


Figure 9 - Case 1

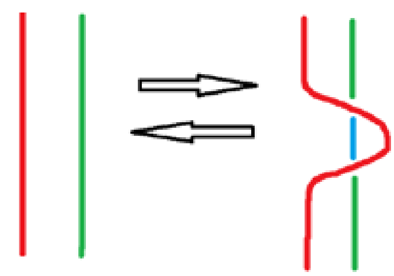


Figure 10 - Case 2

RM-3

There are 3 ways to colour the strands of Reidemeister Move 3 such that all the crossings satisfy the conditions of tri-colourability. In each of these 3 cases, it is possible to recolour the new crossings after the move to preserve tri-colourability.

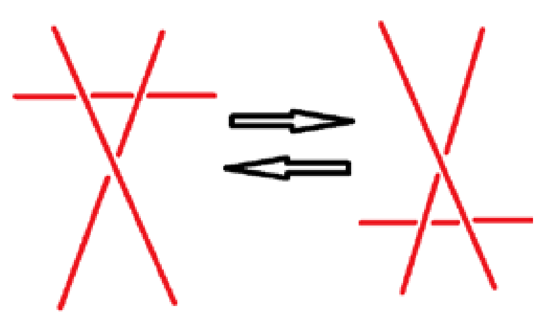


Figure 11 - Case 1

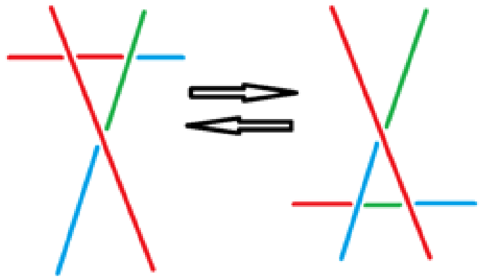


Figure 12 - Case 2

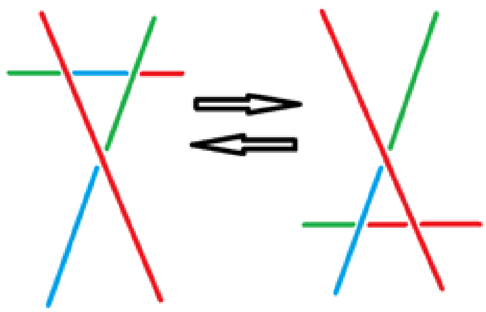


Figure 13 - Case 2

As we see from the figures, tri-colourability is always preserved after making any Reidemeister moves. Hence, it follows that if one projection of a knot is tri-colourable, all projections of that knot are tri-colourable. Therefore, tri-colourability is an invariant for any given knot; a knot is either tri-colourable or not.

Note: Another way to phrase the above is that if even one projection of a knot is not tri-colourable, no projection of the knot is tri-colourable. This should be clear since Reidemeister moves preserve tri-colourability.

Differentiating the unknot from the Trefoil

Figure 14

Until tri-colourability we had no definite way to prove that the two arguably simplest knots – the unknot and the trefoil – are actually different. When we look at them and try to manipulate one into the other, it seems intuitively clear that they are not the same; but that is not a rigorous proof. But we now have a tool – an invariant – that we can use. In the projection of the unknot above, there is only one strand – so we cannot colour it using two or three colours, which is a necessary condition for tri-colourability. Hence the unknot is not tri-colourable. In the picture of the trefoil above (on the right), we can see that it is tri-colourable. Hence, we have a difference between the two knots. Since the trefoil knot is tri-colourable whereas the unknot is not, it follows that the two knots are different.

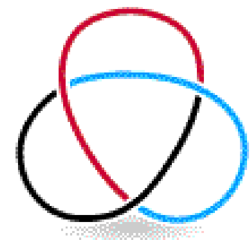


Figure 15

Is It Enough?

Obviously tri-colourability has its uses, since it allows us to definitively differentiate between two knots, something we haven't been able to do until now. But if we dig a little deeper, we see it doesn't help us much more. Tri-colourability is a Boolean invariant – a knot either has the property or it doesn't. Unfortunately, this means there aren't a lot of knots we can differentiate as yet; we just have two categories – knots which are tri-colourable, and those which aren't. Within those categories, we have no way to prove two knots are different. Well, the notion of an invariant helped us before – maybe we just need a more versatile one; maybe a function that has more outputs than just Yes or No (which is what tri-colourability gave us) – maybe even one that is unique to each and every knot.

More Invariants

In an effort to come up with more ways to differentiate knots, mathematicians have defined more and more invariants, some of which I will describe below:

- The *unknotting number* is the minimum number of crossings you have to exchange to change a knot into the unknot. By exchanging a crossing, we mean that at that particular crossing, we change the overcrossing into an undercrossing and vice-versa.

For example, the unknotting number of the trefoil knot is one:

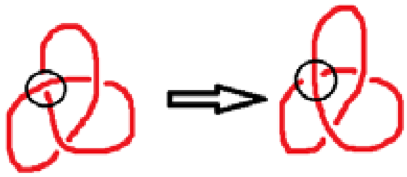


Figure 16

The picture on the right is actually the unknot.

- A knot is *alternating* if there exists at least one projection where the string alternates between crossing over and under.

For example, the trefoil and the figure 8 knot are both alternating knots because if you trace the regular projections of this knot given at the beginning of this article, you will find that the knots alternate over-crossings and under-crossings throughout the entire knot.

- The *crossing number* of a knot is the minimum number of crossings in any projection of a knot.

The crossing number of the unknot is 0. The crossing number of the trefoil is 3. Keep in mind that what this means is that although there are clearly projections of the trefoil knot that can be drawn with more than 3 crossings, the minimum number of crossings that have to be in any projection of the trefoil is 3. Therefore, there is no projection of the trefoil knot that has less than 3 crossings.

- The stick number of a knot is the minimum number of 'sticks' (straight lines) required to make the knot.

The *stick number* of the unknot is 3, and the stick number of the trefoil is 6 as shown below.

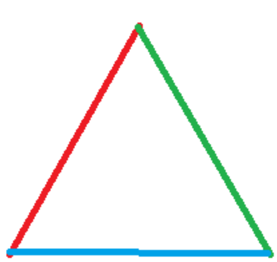


Figure 17

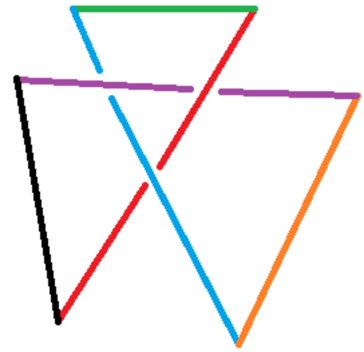


Figure 18

These are but a few of the invariants created in order to differentiate between knots.

The search is still on …

While these invariants are extremely useful in proving knots were different, they are not enough. None of these invariants is unique to each knot – for example, the unknotting number of at least 6 to 7 knots is 1. In order to find something unique to each knot, mathematicians tried to devise something called a knot polynomial – which is an expression in variables that describes the knot. In 1928, the ***Alexander polynomial*** was introduced, the first polynomial of its kind. Later, in 1969, Conway devised the ***Conway polynomial***, which is much simpler to compute than the Alexander Polynomial. In 1984, a few months after the invention of something called the ***Jones polynomial***, a group of six people collectively made the ***HOMFLY polynomial***, which is a superset of all other polynomials made so far. While the HOMFLY polynomial broke ground in this field, the amazing yet frustrating truth is that it is still not enough. Two knots have been discovered having the same HOMFLY polynomial, yet it has been proved using other methods that they are different knots. So, as of now, there is still no definitive answer as to how to prove that two knots are different.

As mentioned earlier, knot theory is a relatively new subject. There is a lot more research that needs to be done. But what we do know is that it is a very interesting subject – and who knows? Maybe you could discover a way to actually prove two knots are different.

References

1. Colin C. Adams, The Knot Book: *An Elementary Introduction to the Mathematical Theory of Knots*
2. Charles Livingston, *Knot Theory* (Mathematical Association of America Textbooks)
3. Richard H. Crowell and Ralph H. Fox, *Introduction to Knot Theory*. Chapter 1 (Dover Books on Mathematics)

Acknowledgements

Some of the diagrams in the article have been taken from the following websites:

1. http://www.wikiwand.com/en/Unknot
2. https://commons.wikimedia.org/wiki/File:Blue_Trefoil_Knot.png
3. https://collegemathteaching.wordpress.com/tag/knot-theory/
4. http://xingyuyang.blogspot.in/2016/02/reidemeister-moves.html
5. https://en.wikipedia.org/wiki/Reidemeister_move#/media/File:Reidemeister_move_2.png
6. https://en.wikipedia.org/wiki/Reidemeister_move#/media/File:Reidemeister_move_3.png

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FILLING A CIRCLE

Rahul tells Devang:

'Imagine you have a circular table and a large number of circular coins. We take turns to place these coins on the table and the player who is unable to find a space to place a coin loses.'

Is there a sure way to win this game?

Does your win depend on whether you are the first player?

Solution on page.......63.......