Features

QUADRILATERALS with Perpendicular Diagonals

SHAILESH SHIRALI

In this article, we study a few properties possessed by any quadrilateral whose diagonals are perpendicular to each other. A four-sided figure possessing such a property is known as an ***ortho-diagonal quadrilateral***. Many special four-sided shapes with which we are familiar have this property: *squares*, *rhombuses* (where all four sides have equal length) and *kites* (where two pairs of adjacent sides have equal lengths). It may come as a surprise to the reader to find that such a simple requirement (diagonals perpendicular to each other) can lead to so many elementary and pleasing properties.

First Property: Sums of squares of opposite sides

Theorem 1. *If the diagonals of a quadrilateral are perpendicular to each other, then the two pairs of opposite sides have equal sums of squares.*

Let the quadrilateral be named $ABCD$. (See Figure 1.) Then the theorem asserts the following: *If* $AC \perp BD$, *then* $AB^2 + CD^2 = AD^2 + BC^2$. Here is a proof. Denoting by O the point where AC meets BD, we have, by the Pythagorean theorem:

$$\begin{cases} AB^2 = AO^2 + BO^2, & CD^2 = CO^2 + DO^2, \\ AD^2 = AO^2 + DO^2, & BC^2 = BO^2 + CO^2, \end{cases} \tag{1}$$

and we see that

$$AB^2+CD^2 = AO^2+BO^2+CO^2+DO^2 = AD^2+BC^2. \quad \square$$

Keywords: *Quadrilateral, perpendicular, diagonals, ortho-diagonal, cyclic quadrilateral, midpoint parallelogram, midpoint rectangle, midpoint circle, maltitude, eight-point circle, indeterminate*

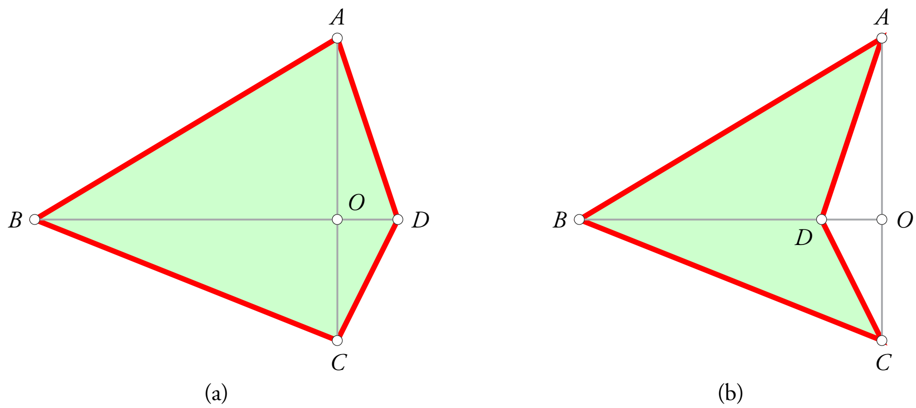


Figure 1

Observe that the conclusion remains true even if the quadrilateral is not convex and the diagonals meet as in Figure 1 (b). The same proof works for both figures.

What is more striking is that the proposition has a converse:

Theorem 2. *If the two pairs of opposite sides of a quadrilateral have equal sums of squares, then the diagonals of the quadrilateral are perpendicular to each other.*

That is, if $AB^2 + CD^2 = AD^2 + BC^2$, then $AC \perp BD$. We urge the reader to try proving this before reading on.

Our proof uses the generalized version of the theorem of Pythagoras: *In $\triangle ABC$, the quantity a^2 is less than, equal to, or greater than $b^2 + c^2$, in accordance with whether $\measuredangle A$ is less than, equal to, or greater than* $90°$.

We wish to show that if $AB^2 + CD^2 = AD^2 + BC^2$, then $\measuredangle AOB = 90°$. Our approach will be to show that $\measuredangle AOB$ cannot be either acute or obtuse; this leaves only one possibility, the one we want. (Euclid was fond of this approach. Several proofs in The Elements are presented in this style. Sherlock Holmes too was fond of this principle! Holmes enthusiasts will remember his unforgettable sentence, "How often have I said to you that when you have eliminated the impossible, whatever remains, however improbable, must be the truth?") To start with, suppose that $\measuredangle AOB$ is acute. Then $\measuredangle COD$ too is acute, and $\measuredangle BOC$ and $\measuredangle DOA$ are obtuse. By the generalized version of the PT, we have:

$$\begin{cases} AB^2 < AO^2 + BO^2, & CD^2 < CO^2 + DO^2, \\ AD^2 > AO^2 + DO^2, & BC^2 > BO^2 + CO^2. \end{cases} \tag{2}$$

By addition we get the following double inequality:

$$AB^2 + CD^2 < AO^2 + BO^2 + CO^2 + DO^2 < AD^2 + BC^2, \tag{3}$$

implying that $AB^2 + CD^2 < AD^2 + BC^2$. This contradicts what we were told at the start: that $AB^2 + CD^2 = AD^2 + BC^2$. *So our supposition that $\measuredangle AOB$ is acute must be wrong; $\measuredangle AOB$ cannot be acute.*

If we suppose that $\measuredangle AOB$ is obtuse, we get $AB^2 + CD^2 > AD^2 + BC^2$. This too contradicts what we were told at the start and must be discarded. *$\measuredangle AOB$ cannot be obtuse.*

The only possibility left is that $\measuredangle AOB$ is a right angle; i.e., that AC and BD are perpendicular to each other. Which is the conclusion we were after. $\square$

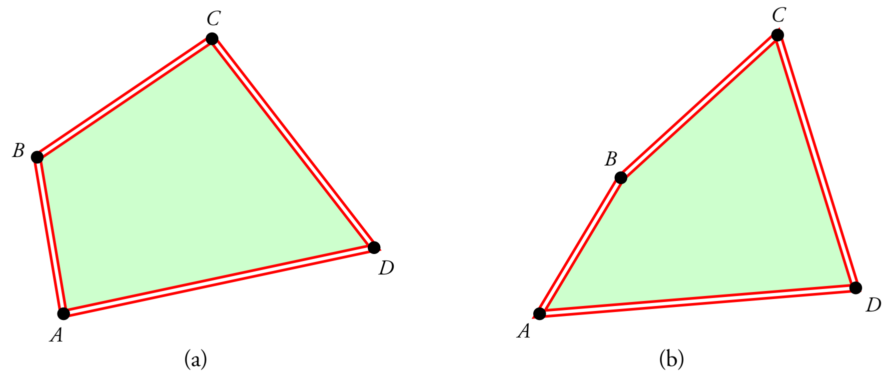


Figure 2. A pair of jointed quadrilaterals with the same side lengths

A proof using vectors. Students of classes 11 and 12 may be interested in seeing that there is a vector proof of Theorems 1 and 2; both theorems are proved at the same time.

Given a quadrilateral $ABCD$, we denote the vectors **AB**, **BC**, **CD** by **a**, **b**, **c** respectively; then $\mathbf{AD} = \mathbf{a} + \mathbf{b} + \mathbf{c}$. We now have:

$$AB^2 = \mathbf{a} \cdot \mathbf{a}, \quad BC^2 = \mathbf{b} \cdot \mathbf{b}, \quad CD^2 = \mathbf{c} \cdot \mathbf{c}, \tag{4}$$

$$\begin{aligned} AD^2 &= (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) \\ &= \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{c} + 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}). \end{aligned} \tag{5}$$

Hence:

$$\begin{aligned} AD^2 + BC^2 - AB^2 - CD^2 &= 2(\mathbf{b} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}) \\ &= 2(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} + \mathbf{c}) = 2\mathbf{AC} \cdot \mathbf{BD}. \end{aligned} \tag{6}$$

Hence $AD^2 + BC^2 - AB^2 - CD^2 = 0$ if and only if $AC \perp BD$. □

Second Property: Rigidity

Theorem 2 has a pretty consequence related to the notion of *rigidity* of a polygon.

We know that a triangle is rigid: given three lengths which satisfy the triangle inequality (i.e., the largest length is less than the sum of the other two lengths), there is just one triangle having those lengths for its sides. So its shape cannot change. If we make a triangle using rods for sides, joined together at their ends using nuts and bolts, the structure is **rigid** and **stable**; it will not lose shape when subjected to pressure.

But a quadrilateral made this way is not rigid; if at all one can make a quadrilateral using four given lengths as its sides (this requires that the largest length is less than the sum of the other three lengths), one can 'push' it inwards or 'pull' it outwards and so deform its shape. There are thus infinitely many distinct quadrilaterals that share the same side lengths. Figure 2 illustrates this property.

Now let a 'jointed quadrilateral' be formed in this manner using four given rods. Then, as just noted, we can deform its shape by applying pressure at the ends of a diagonal, inward or outward. Suppose it happens that in some position the diagonals of the quadrilateral are perpendicular to each other. *Then this property is never lost, no matter how we deform the quadrilateral. The diagonals always remain perpendicular to each other.*

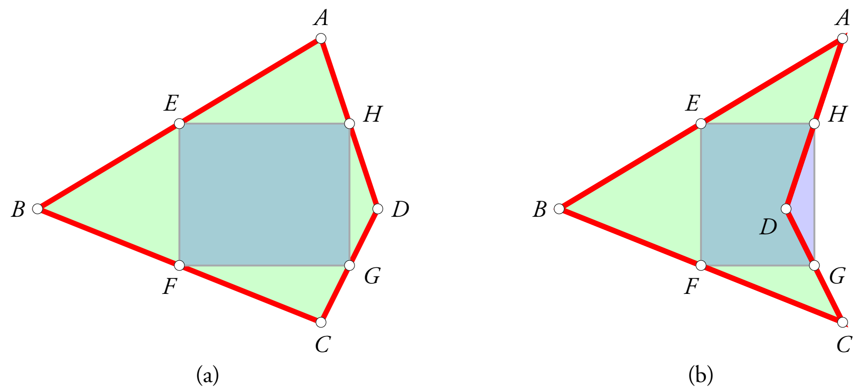


Figure 3

Third Property: Midpoint rectangle
This property will not come as a surprise to students in classes 9 and 10, who will be familiar with the midpoint theorem for triangles. It is illustrated in Figure 3.

Given a quadrilateral $ABCD$, let the midpoints of its sides AB, BC, CD, DA be E, F, G, H respectively. Then, by the Midpoint Theorem, segments EF and HG are parallel to diagonal AC, and segments FG and EH are parallel to diagonal BD. Hence the quadrilateral $EFGH$ is a parallelogram; it is called the **midpoint parallelogram** of $ABCD$. (This result by itself is known as **Varignon's Theorem**. It is true for any quadrilateral.) Hence, if the diagonals of $ABCD$ are perpendicular to each other, all the angles of $EFGH$ are right angles. This yields Theorem 3.

Theorem 3. *If the diagonals of the quadrilateral are perpendicular to each other, then its midpoint parallelogram is a rectangle.*

As earlier, the result remains true if the quadrilateral is non-convex, as in Figure 3 (b). The midpoint parallelogram may now be called the **midpoint rectangle**.

The converse of Theorem 3 is also true, namely:

Theorem 4. *If the midpoint parallelogram of a quadrilateral is a rectangle, then the diagonals of the quadrilateral are perpendicular to each other.*

We omit the proof, which is quite easy.

Fourth Property: Midpoint Circle
A rectangle is a special case of a cyclic quadrilateral. Hence, for any quadrilateral whose diagonals are perpendicular to each other, there exists a circle which passes through the midpoints of its four sides. Now a line which intersects a circle must do so again at a second point (possibly coincident with the first point of intersection, which would be a case of tangency). So the midpoint circle must intersect each of the four sidelines again, giving rise to four special points; see Figure 4. What are these points?

Figure 4 depicts the situation. The midpoint of the sides are E, F, G, H respectively, and circle ($EFGH$) intersects the four sides at points I, J, K, L. (It so happens that in this particular figure, L has coincided with H. However, this will obviously not happen in general.) Since the line segments EG and FH (which are diagonals of the rectangle $EFGH$) are diameters of the circle, it follows that I is the foot of the perpendicular from G to line AB; J is the foot of the perpendicular from H to line BC; K is the foot of the perpendicular from E to line CD; and L is the foot of the perpendicular from F to line DA. (This follows from the theorem of Thales: "the angle in a semicircle is a right angle.") So we have now identified what

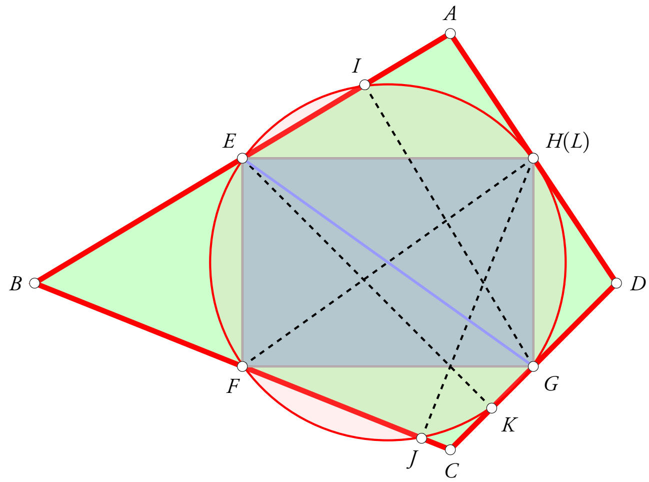


Figure 4

the four new points are: they are the feet of the perpendiculars drawn from the midpoints of the sides to the opposite sides. The circle passing through these eight points is referred to as the **eight-point circle** of the quadrilateral (in analogy with the much better-known nine point circle of a triangle).

These four perpendiculars we have drawn (from the midpoints of the sides to the opposite sides) are called the **maltitudes** of the quadrilateral. Clearly, this construction can be done for any quadrilateral. The theorem enunciated above can now be written in a stronger form:

Theorem 5. *If the diagonals of the quadrilateral are perpendicular to each other, then the midpoints of the four sides and the feet of the four maltitudes lie on a single circle.*

The converse of this statement is also true:

Theorem 6. *If the midpoints of the four sides and the feet of the four maltitudes lie on a single circle, then the diagonals of the quadrilateral are perpendicular to each other.*

Fifth Property: Brahmagupta's theorem

In this section, we study an interesting property of a cyclic quadrilateral whose diagonals are perpendicular to each other; thus, we have imposed an additional property on the quadrilateral, namely, that it is *cyclic*. The property in question was first pointed out by the Indian mathematician Brahmagupta (seventh century AD).

Figure 5 depicts the property. The cyclic quadrilateral in question is $ABCD$; its diagonals AC and BD are perpendicular to each other. The midpoints of its sides are E, F, G, H, and $EFGH$ is a rectangle. The circle through E, F, G, H intersects the four sides again at points I, J, K, L respectively. If we draw the segments EK, FL, GI, HJ respectively, we find that they all pass through the point of intersection of AC and BD. So we have six different segments passing through a common point. Not only that, but we also have $EK \perp CD$, $FL \perp DA$, $GI \perp AB$ and $HJ \perp BC$! Indeed a beautiful result. We state the result formally as a theorem (see [3]):

Theorem 7 (Brahmagupta)**.** *If a quadrilateral is both cyclic and ortho-diagonal, then the perpendicular to a side from the point of intersection of the diagonals bisects the opposite side.*

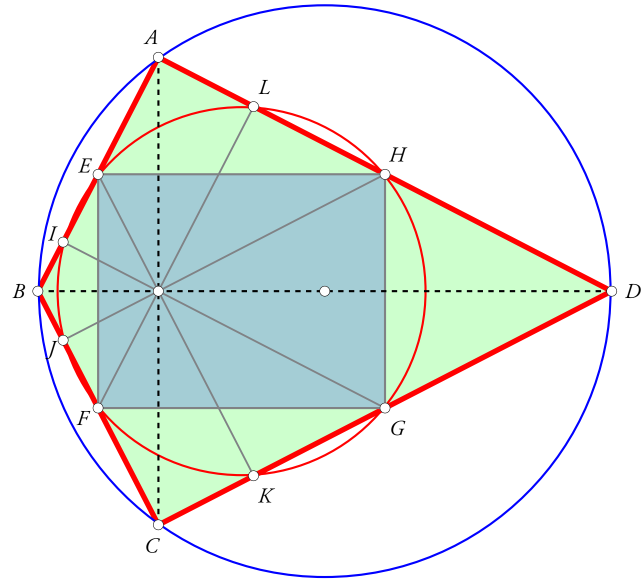


Figure 5

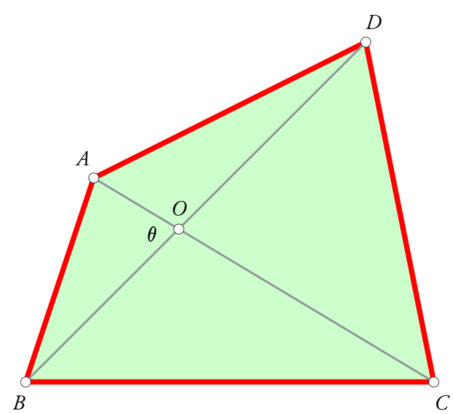


Figure 6

Sixth Property: Area

We conclude this article with a discussion concerning the **area** of a quadrilateral. We examine the following question: *If we know the four sides of a quadrilateral and the angle between its two diagonals, can we find the area of the quadrilateral?* We shall show that the answer in general is 'Yes.' But it is not an unqualified Yes!—there is an unexpected twist in the tale.

Let the quadrilateral be named $ABCD$, let the sides AB, BC, CD, DA have lengths a, b, c, d respectively, let the point of intersection of the diagonals AC and BD be O, and let it be specified that the angle between the diagonals has measure θ. We need to find a formula for the area k in terms of a, b, c, d, θ. Using the sine formula for the area of a triangle, it is easy to show that the area of the quadrilateral is

$$k = \frac{1}{2} AC \times BD \times \sin\theta. \tag{7}$$

We had shown in Section I that $AD^2 + BC^2 - AB^2 - CD^2 = 2\mathbf{AC} \cdot \mathbf{BD}$, i.e.,

$$\left|AD^2 + BC^2 - AB^2 - CD^2\right| = 2\,AC \times BD \times |\cos\theta|\,. \tag{8}$$

(We have used absolute value signs here as the signs do not matter any longer at this stage; we are only interested in the absolute magnitudes.) Hence:

$$AC \times BD = \frac{|AD^2 + BC^2 - AB^2 - CD^2|}{2\ |\cos\theta|}. \tag{9}$$

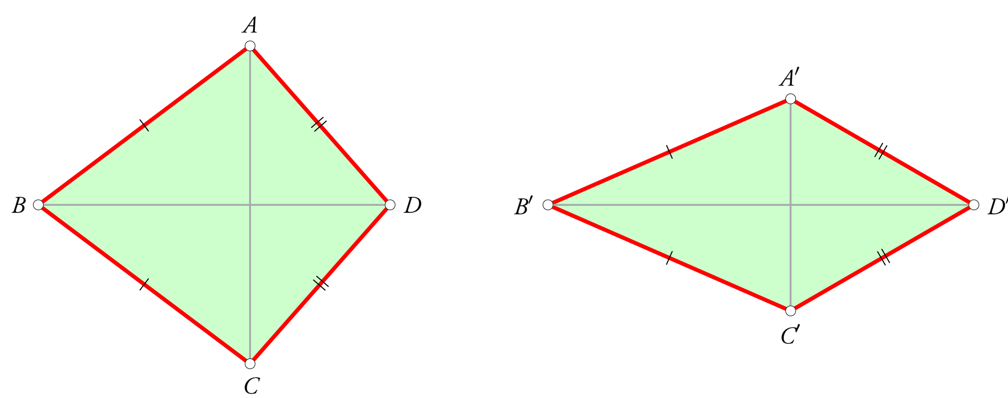
This yields the desired formula for the area of the quadrilateral:

$$k = \frac{|AD^2 + BC^2 - AB^2 - CD^2| \times |\tan\theta|}{4}. \tag{10}$$

At this point, we uncover something quite fascinating. Suppose the quadrilateral in question is of the type studied here; i.e., its diagonals are perpendicular to each other. Then in the above formula, we encounter an indeterminate form! For, in the numerator, when $\theta = 90^\circ$, we see the product $0 \times \infty$. Examining the situation geometrically, we realise that this is not just a numerical quirk; the area really cannot be determined!

The comments made in Section II should make this clear; we pointed out that a quadrilateral is not rigid; but that as it changes shape, the property that its diagonals are perpendicular to each other is invariant. If the property is true in one particular configuration, it always remains true. We see directly in this situation that the area can assume a whole continuum of values, implying that it is indeterminate.

The case of a kite. Figure 7 displays a particular case of this which is easy to grasp: when adjacent pairs of sides have equal length (i.e., the figure is a kite).



(a)

(b)

Figure 7

Figures $ABCD$ and $A'B'C'D'$ (which are kites, with $AD = CD = A'D' = C'D'$ and $AB = BC = A'B' = B'C'$) have unequal area. By further squashing the kite along the vertical diagonal, we can make its area as small as we wish. The limiting case in this direction would be that of a **degenerate quadrilateral**, i.e., one with zero area.

The case of a rhombus. As noted in [2], the phenomenon described above may be visualized still more easily in the case of a rhombus. Let the rhombus have side a, and let its angles be α and $180^\circ - \alpha$ where $0^\circ < \alpha \leq 90^\circ$ (see Figure 8). Then the area of the rhombus is equal to $a^2 \sin\alpha$.

Since α can assume any value between 0° and 90°, it follows that the area of the rhombus can assume any value between 0 and a^2.

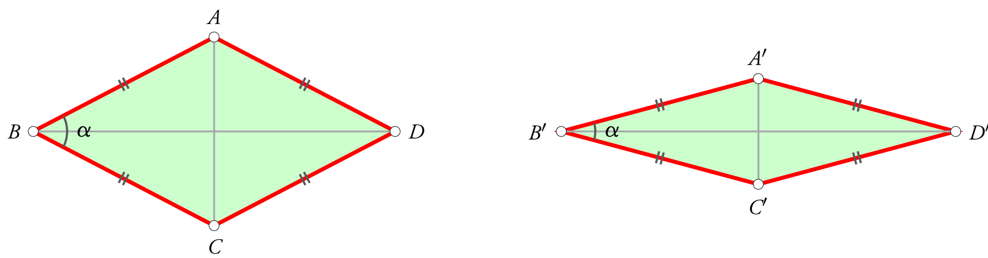


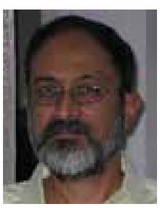
Figure 8

References

1. Martin Josefsson, *Characterizations of Orthodiagonal Quadrilaterals*. http://forumgeom.fau.edu/FG2012volume12/FG201202.pdf

2. Wikipedia, *Orthodiagonal quadrilateral*. https://en.wikipedia.org/wiki/Orthodiagonal_quadrilateral

3. Wikipedia, *Brahmagupta theorem*. https://en.wikipedia.org/wiki/Brahmagupta_theorem



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