ClassRoom

Divisibility by PRIMES

VINAY NAIR

In school we generally study divisibility by divisors from 2 to 12 (except 7 in some syllabi). In the case of composite divisors beyond 12, all we need to do is express the divisor as a product of coprime factors and then check divisibility by each of those factors. For example, take the case of 20; since 20 = 4 × 5 (note that 4 and 5 are coprime), it follows that a number is divisible by 20 if and only if it is divisible by 4 as well as 5. It is crucial that the factors are coprime. For example, though 10 × 2 = 20, since 10 and 2 are *not* coprime, it cannot be asserted that if a number is divisible by both 10 and 2, then it will be divisible by 20 as well. You should be able to find a counterexample to this statement.

Divisibility tests by primes such as 7, 13, 17 and 19 are not generally discussed in the school curriculum. However, in *Vedic Mathematics* (also known by the name "High Speed Mathematics"; see Box 1), techniques for testing divisibility by such primes are discussed, but without giving any proofs. In this article, proofs of these techniques are discussed.

Divisibility test for numbers ending with 9

The author of Vedic Mathematics gives divisibility tests for the divisors 19, 29, 39, ... (These are all numbers ending with 9.) These tests all have the same form. We add 1 (according to the sutra '*Ekadhikena Purvena*' which means: "by one more than the previous") to 19, 29, 39, ... Doing so, we get 20, 30, 40, ... The **osculators** for 19, 29, 39 are then 2, 3, 4, respectively (the digits left in 20, 30, 40 after ignoring the 0). The manner in which the osculator is to be used is explained below.

Keywords: Divisibility, divisor, osculator, Vedic mathematics

Notation. Any positive integer N can be written in the form $10a + b$, where b is the units place digit (so $b \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$) and a is an integer. We refer to a as 'the rest' of the number (after deleting the units digit). For example, if $N = 2356$, then $b = 6$ and $a = 235$. If d is the divisor, we denote the osculator for d by k.

Example 1. We check if 114 is divisible by 19. We follow these steps:

Step 0: This is the "initialisation step." The divisor is $d = 19$; its osculator is $k = 2$. The initial values of N, b, a are $N = 114$ (the given number), $b = 4$ (units digit), $a = 11$ ('rest of the number'). The values of N, b, a will get updated in subsequent steps, as shown below.

Step 1: Compute c using the formula c = units place digit of $N \times$ the osculator for 19, i.e., $c = bk$. We get $c = 4 \times 2 = 8$.

Step 2: Add the number c obtained in Step 1 to a, i.e., to the rest of N after ignoring the digit b considered for the operation in Step 1; so we do $a + c = 11 + 8 = 19$. *This is the updated value of N. We now update the values of a and b, using the updated value of N.*

Step 3: Check mentally if the updated value of N is divisible by 19. If yes, then we conclude that the original number is divisible by 19. If we are not sure, then repeat Steps 1–2–3 till we get a number which we know mentally is divisible (or not divisible) by 19. Please note that we must update the values of N, a, b each time we go through the cycle.

Example 2. We check whether 2356 is divisible by 19.

Step 0: Initialisation: $d = 19$; $k = 2$; $N = 2356$ (the given number), $b = 6$ (units digit), $a = 235$ ('rest of the number').

Step 1: Compute $c = bk = 6 \times 2 = 12$.

Step 2: Compute $a + c = 235 + 12 = 247$. This is now the updated value of N. So the updated values are: $N = 247$, $b = 7$, $a = 24$.

Step 3: Is 247 divisible by 19? As we are not sure, we repeat Steps 1–2.

Step 1: Compute $c = bk = 7 \times 2 = 14$.

Step 2: Compute $a + c = 24 + 14 = 38$.

About Vedic Mathematics

Vedic Mathematics is a way of doing calculations using 16 sutras (aphorisms) in Sanskrit discovered by Swami Bharati Krishna Tirtha, a saint in the Shankaracharya order, during 1911–1918 CE. For tests of divisibility, the founder uses something called *osculators*.

Note from the editor. Readers could refer to the following Wikipedia reference [1]. Its opening paragraph is the following:

> Vedic Mathematics is a book written by the Indian Hindu priest Bharati Krishna Tirthaji and first published in 1965. It contains a list of mental calculation techniques claimed to be based on the Vedas. The mental calculation system mentioned in the book is also known by the same name or as "Vedic Maths". Its characterization as "Vedic" mathematics has been criticized by academics, who have also opposed its inclusion in the Indian school curriculum.

References

i. Wikipedia, "Vedic Mathematics (book)", https://en.wikipedia.org/wiki/Vedic_Mathematics_(book)

Step 3: Is 38 divisible by 19? Yes. Hence 2356 is divisible by 19.

Example 3. We check whether 1234 is divisible by 19.

Step 0: Initialisation: $d = 19$; $k = 2$; $N = 1234$ (the given number), $b = 4$ (units digit), $a = 123$ ('rest of the number').

Step 1: Compute $c = bk = 4 \times 2 = 8$.

Step 2: Compute $a + c = 123 + 8 = 131$. Now $N = 131$, $b = 1$, $a = 13$.

Step 3: Is 131 divisible by 19? As we may not be sure not sure of this, we repeat Steps 1–2–3.

Step 1: Compute $c = bk = 1 \times 2 = 2$.

Step 2: Compute $a + c = 13 + 2 = 15$.

Step 3: Is 15 divisible by 19? No. Hence 1234 is not divisible by 19.

A doubt. One may not know when to stop the process. Steps 1–2–3 have to be continued till we come across a number which is small enough that we know directly whether it is or is not a multiple of 19. Consider the number 1121. After the first iteration, we get 114. If one knows that 114 is divisible by 19, then the process can be stopped here. If one does not know it, then the steps can be continued from 114 as in Example 1.

The process for 29, 39, 49, 59, . . . is the same; the osculators are 3, 4, 5, 6, . . . respectively.

Rationale behind the process

Consider the number 114. When we are doing $4 \times 2 = 8$ and adding it to 11, getting $11 + 8 = 19$, what we are 'actually' doing is to compute 110 + 80. But now we have:

$$110 + 80 = 114 - 4 + 80 = 114 + 76 = 190.$$

In effect, therefore, we have added 76 to the original number. The significant point here is that 76 is a multiple of 19. Since both 190 and 76 are multiples of 19, the original number 114 too must be a multiple of 19.

Instead of 114, think of a number ending with the digit 1; e.g., 171. When we multiply the units digit with the osculator 2 and add the product to the rest of the number, we are actually adding 20 and subtracting 1 (as we ignore the units digit 1). So we are actually adding $20 - 1 = 19$ to the number; and 19 is a multiple of 19. Similarly, when the units digit is another digit, say 5, the calculation is now $5 \times 2 = 10$. This means we are adding $100 - 5 = 95$, which again is a multiple of 19.

Similarly for the tests of divisibility by 29, 39, 49, ... Essentially the same logic works in each case.

Finding osculators for other prime numbers

Once the osculators are known, the process of checking divisibility remains the same. We only need to know how to find the osculator.

Consider a prime number like 7 which does not end with the digit 9. In such cases, we consider a multiple of 7 that ends with 9. The smallest such multiple is 49. We use the sutra *Ekadhikena purvena* and add 1 to it to get 50. Ignore the 0 and consider the remaining part of 50 as the osculator; we get 5. Hence 5 is the osculator for checking divisibility by 7.

If we look carefully, the process is simple. By taking the osculator as 5, we are really checking if the number is divisible by 49. Since the number is divisible by 49, it is definitely divisible by 7.

In the same way, the osculators for 13, 17, 23, . . . can be obtained; they are 4, 12, 7, respectively.

Divisibility test for numbers ending with 1

Let's take the case of divisibility by numbers like 21, 31, 41, ... Here, instead of adding 1 to the numbers (as in the case of divisibility by numbers ending with the digit 9), we subtract 1; this is the instruction given in the sutra *Ekanyunena Purvena* ("by one less than the previous"). So the numbers to be considered for divisibility by 21, 31, 41 are 20, 30, 40, respectively. As earlier, we consider the digits other than 0 to be the osculator, i.e., for 20, 30 and 40, the osculators are 2, 3 and 4, respectively. After this we follow the same process as in the case for divisibility by numbers ending with 9; but now we ***subtract*** rather than add. An example will make this clear.

Example 4. We check whether 441 is divisible by 21.

Step 0: Initialisation: $d = 21$; $k = 2$; $N = 441$ (the given number), $b = 1$ (units digit), $a = 44$ ('rest of the number').

Step 1: Compute $c = bk = 1 \times 2 = 2$.

Step 2: Compute $a - c = 44 - 2 = 42$. Now $N = 42$, $b = 2$, $a = 4$.

Step 3: Is 42 divisible by 21? Yes. Hence 441 is divisible by 21.

Rationale. The rationale is the same as earlier. When we multiply the units digit 1 by 2 and subtract from 44, we are actually doing $440 - 20 = 441 - 1 - 20 = 441 - 21$.

So we have subtracted 21 from the number. Since $441 - 21 = 420$, and 420 is a multiple of 21, the original number 441 is a multiple of 21.

If the units digit of a number happens to be 3, and if we multiply by 2 and subtract from the rest of the number, the actual process happening is $3 \times 2 = 6$. When the units digit 3 is ignored, we are subtracting 3 from the original number. When 6 is subtracted from the rest of the number, because of decimal place value system, we are actually subtracting 60. Since 3 is ignored, we are subtracting 3 as well, resulting in a total subtraction of 63. Since 63 is a multiple of 21, if the final number is a multiple of 21, the original number itself must be a multiple of 21. Likewise for any other units digit.

Remark. To find the osculator for 7, we can use the osculator of 49, as seen above. But we can also use the osculator of 21, since 21 is a multiple of 7. Thus there will be two osculators for every prime number depending on whether we use multiplication and addition or multiplication and subtraction. Given below is a table of some divisors and their two osculators.

Divisor	7	13	17	23	27	37
Multiple ending with 9	49	39	119	69	189	259
Corresponding osculator	5	4	12	7	19	26
Multiple ending with 1	21	91	51	161	81	111
Corresponding osculator	2	9	5	16	8	11

A study of the table reveals that *the sum of the two osculators of a divisor is the divisor itself*. For example, the osculators for 13 are 4 and 9, and $4 + 9 = 13$. One has the liberty to choose whichever seems more convenient. For example, in the case of 17, if we choose the osculator 12, the calculation is cumbersome. Instead, the osculator 5 makes it easier.

With this approach in Vedic Mathematics, tests of divisibility by any number can be devised.

VINAY NAIR is a co-founder of *Raising a Mathematician Foundation*. He conducts various online and offline programs in different parts of India on exploratory learning in Mathematics and ancient Indian Mathematics. He aspires to create a research mentality in the minds of school children. He can be reached at vinay@sovm.org.