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An investigation with TRIANGLES & CIRCLES

PARTHIV DEDASANIYA

Often concepts as simple as triangles and circles can give rise to very interesting problems. One such problem was given to us by our Math Teacher in class when we were learning about incircles and incentres. When given the problem, it did not occur to me that the problem had to be solved using incentres. Instead, I thought of the problem in a completely different method and was able to solve it.

Problem. Given an arbitrary triangle *ABC*, the problem is to draw three circles with their centres at the three vertices of the triangle, in such a way that each circle touches the other two circles (i.e., is tangent to the other two circles), as in Figure 1.

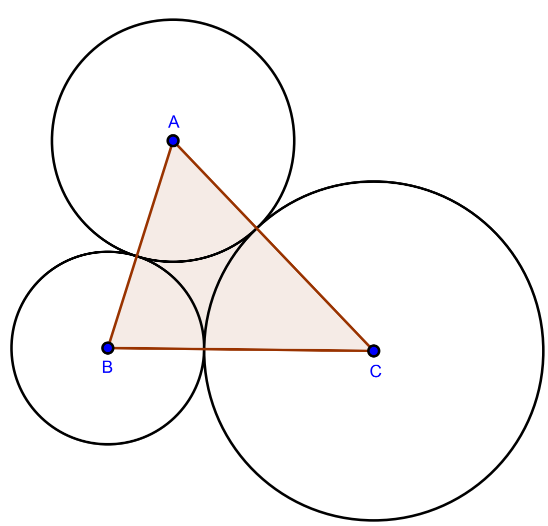


Figure 1

Keywords: Triangle, incircle, incentre

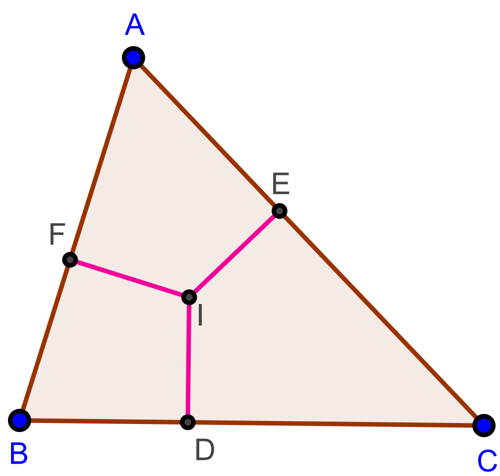


Figure 2

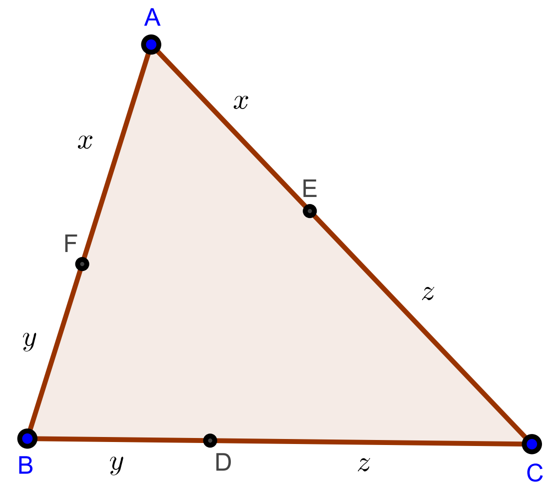


Figure 4

Solution. Here is how the problem was *expected* to be solved. First locate the incentre I of the triangle (this is the point where the internal bisectors of the three angles of the triangle meet). Next, drop perpendiculars ID, IE and IF from I to the sides of the triangle. (See Figure 2; the angle bisectors have not been shown.)

Lastly, draw circles: centred at A and passing through E and F; centred at B and passing through F and D; and centred at C and passing through D and E. These circles are the ones we seek. (See Figure 3.)

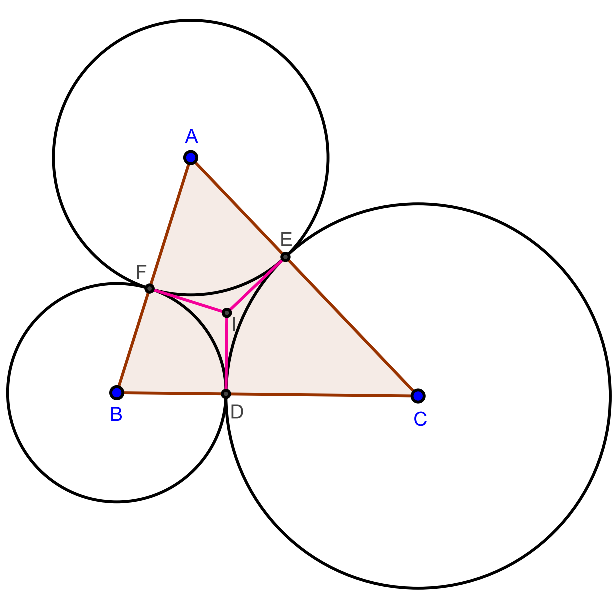


Figure 3

Explanation. As the incentre lies on the bisectors of the three angles of the triangle, it is equidistant from the three sides of the triangle; that is, the segments ID, IE, IF have equal length (Figure 2). From this it is easy to deduce, using triangle congruence, that $BD = BF$, $AF = AE$, $CE = CD$. (Draw the segments IA, IB, IC to see why.) Hence the circles can be drawn as shown in Figure 3.

My solution. I reasoned out a solution in the following way. Let the lengths of the sides BC, CA, AB be a, b, c, respectively, and let the radii of the circles centred at A, B, C be x, y, z respectively. The picture appears as shown in Figure 4. The problem is to find x, y, z respectively.

In the discussion below, I shall assume that side AB is less than side AC in length. This means that $x + y < x + z$, i.e., $y < z$. Mark off a length AU on side AC such that $AU = AB$ (see Figure 5.1). This is possible since $AB < AC$. The length of CU is $z - y$.

Next, locate a point V on side BC such that $CV = CU$ (see Figure 5.2), and locate the midpoint D of segment BV. Then BD is the radius of one of the desired circles, the one centred at B.

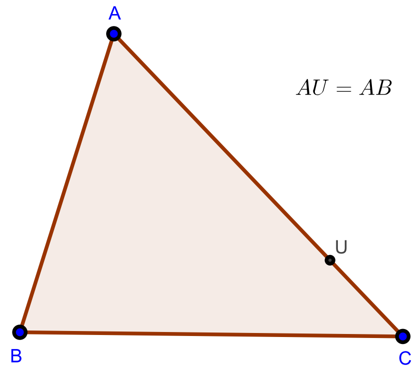


Figure 5.1

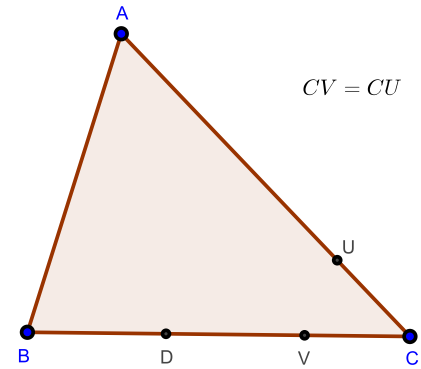


Figure 5.2

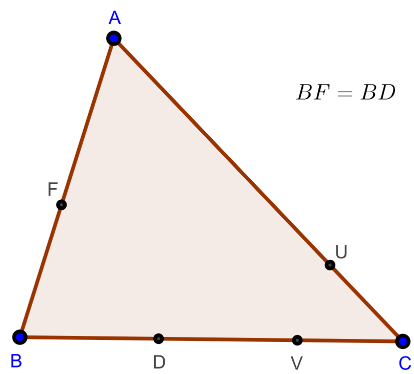


Figure 6.1

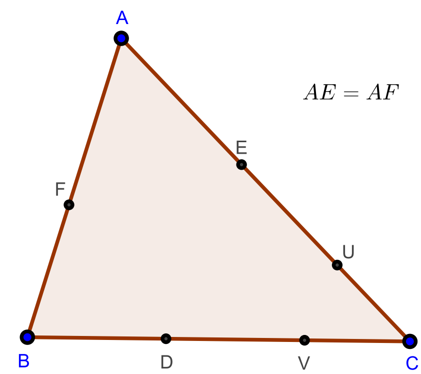
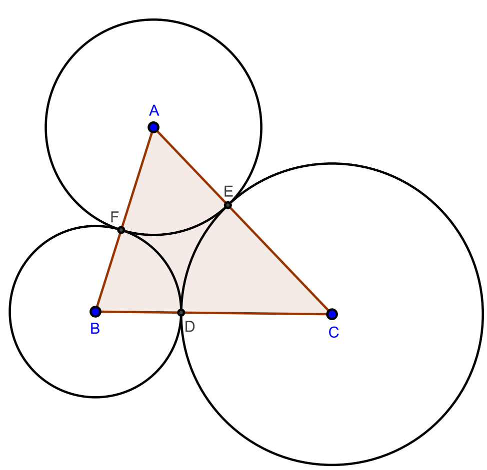


Figure 6.2

Follow this by locating a point F on side AB such that $BD = BF$ (Figure 6.1), and then by locating a point E on side AC such that $AE = AF$ (Figure 6.2). These constructions provide the points D,E,F. Once these three points have been located, the circles can be drawn as earlier (Figure 7).

Explanation. The reason this procedure works is this. Remember that $BC = a$, $CA = b$, $AB = c$. In Figures 5.1, 5.2, 6.1 and 6.2, $AU = c$, hence $CU = b - c = CV$, therefore $BV = a - (b - c) = a - b + c$. This implies that $BD = (a - b + c)/2 = s - b$, where s is the semi-perimeter of the triangle, $s = (a + b + c)/2$. From this we get:

$$AF = c - \frac{a - b + c}{2} = \frac{-a + b + c}{2} = s - a;$$



so $AE = s - a$ as well. Next, notice that

$$CE = b - \frac{-a+b+c}{2} = \frac{a+b-c}{2} = s - c,$$

and also

$$CD = a - \frac{a-b+c}{2} = \frac{a+b-c}{2} = s - c;$$

so $CE = CD$. Therefore we have:

$$BD = BF,\ AF = AE,\ CE = CD,$$

which means that the circle centred at B, with radius BD, passes through F; the circle centred at A, with radius AF, passes through E; and the circle centred at C, with radius CE, passes through D. These three circles are therefore identical with the three circles drawn in the earlier construction (Figure 3). Hence the circles can be drawn as described.

PARTHIV DEDASANIYA is currently in grade 12 (IB) of Kodaikanal International School. He has a passion for solving mathematical puzzles and problems. He has won medals in mathematical Olympiads, including a distinction in the Waterloo test. He enjoys tutoring junior students who face difficulties in mathematics. He hopes to study robotics or nano-technology in college. Parthiv may be contacted at parthiv3215@gmail.com.