FAGNANO'S PROBLEM
Addendum

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The problem treated in the accompanying article is this: Given an arbitrary acute-angled triangle PQR, inscribe within it a triangle ABC, with A on side RP, B on side PQ, and C on side QR, having the smallest possible perimeter. The author establishes, using geometrical arguments, that in the optimal configuration, the following triangle similarities must hold (see Figure 1):

$$\triangle ARC \sim \triangle QBC \sim \triangle ABP \sim \triangle QRP,$$

and then shows, using trigonometry, that these conditions force A, B, C to be the feet of the altitudes of the triangle.

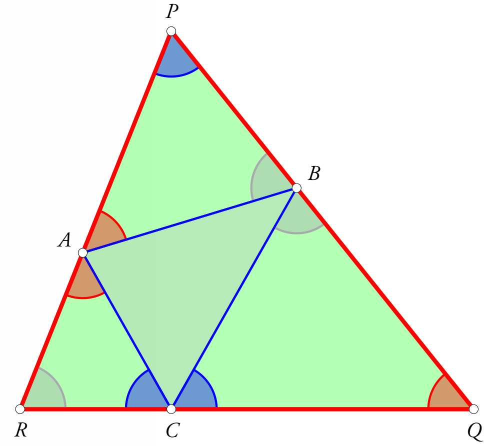
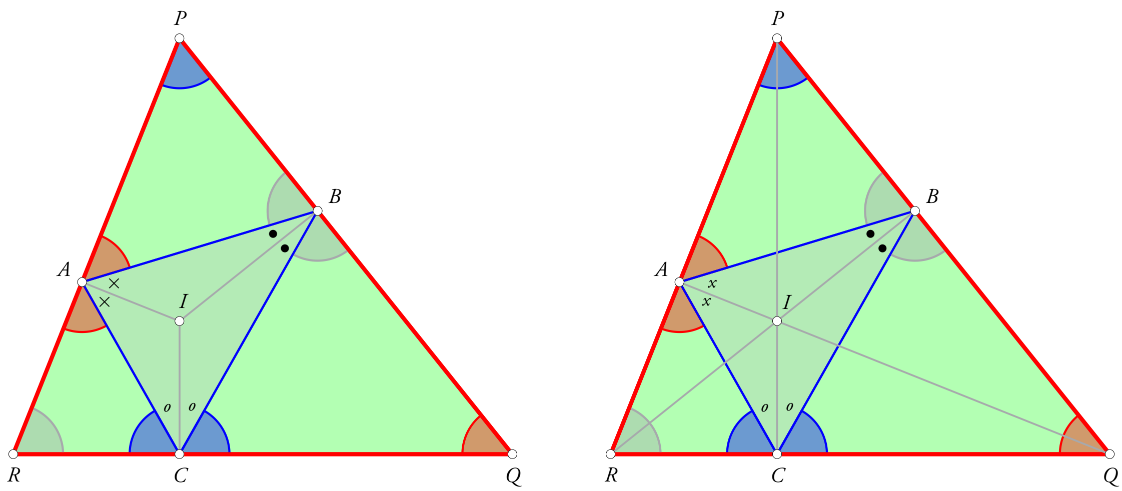


Figure 1. Fagnano's Problem

Here we provide a geometrical proof of this proposition. We also justify the need to impose the condition that triangle PQR should be acute-angled.

Keywords: triangle, acute, obtuse, perpendicular, angle bisector, incentre, excentre, collinear



(a) (b)

Figure 2

Proof of proposition
Construct the internal bisectors of the angles of $\triangle ABC$. The three lines thus constructed meet at the incentre I of $\triangle ABC$; see Figure 2 (a).

It is easy to check, using elementary angle computations, that the sides of $\triangle PQR$ are respectively perpendicular to the three angle bisectors; that is, side QR is perpendicular to the angle bisector CI of $\measuredangle ACB$, and so on. But this implies that the sides of $\triangle PQR$ are respectively the external bisectors of the angles of $\triangle ABC$ (i.e., side QR is the external angle bisector of $\measuredangle ACB$, and so on). This in turn implies that P, Q, R are the ex-centres of $\triangle ABC$. And this in turn implies that P, Q, R lie on the (internal) angle bisectors of $\measuredangle ACB$, $\measuredangle CAB$, $\measuredangle ABC$ respectively. That is, points P, I, C are collinear, as are points Q, I, A and points R, I, B; see Figure 2 (b).

It follows that PC, QA and RB are the altitudes of $\triangle PQR$. This is just what we had set out to prove. $\square$

Why should the triangle be acute angled?
We now justify the need to impose the condition that $\triangle PQR$ should be acute angled. We accomplish this by considering what happens if $\triangle PQR$ is right-angled or obtuse-angled.

Figures 3 (a), 3 (b) and 3 (c) show triangles in each of which the angle at vertex R is successively larger than in Figures 1 and 2; it is getting 'closer' to a right angle, and in the limit, Figure 3 (c), the triangle becomes right-angled at vertex R.

Observe carefully what happens: as $\measuredangle R$ increases, vertices A and C get steadily closer to each other, and in the limit, when the triangle becomes right-angled at vertex R, the two vertices coincide with R. When this happens, $\triangle ABC$ collapses into segment RB. The configuration will now be as depicted in Figure 3 (c). We infer from this that if $\triangle PQR$ is right-angled, then the inscribed triangle with least perimeter is a line segment. (Note that in the limiting situation, segment BR is traced out *twice*, which means that the perimeter of $\triangle ABC$ is twice the length of segment BR.)

It is possible to show directly that if $\triangle PQR$ is right-angled at R and $\triangle DEF$ is inscribed in $\triangle PQR$, then its perimeter cannot be less than twice the length of altitude RB. Let DEF be any inscribed triangle, as in Figure 3 (d). We now perform the following geometrical operations on this figure: we reflect the

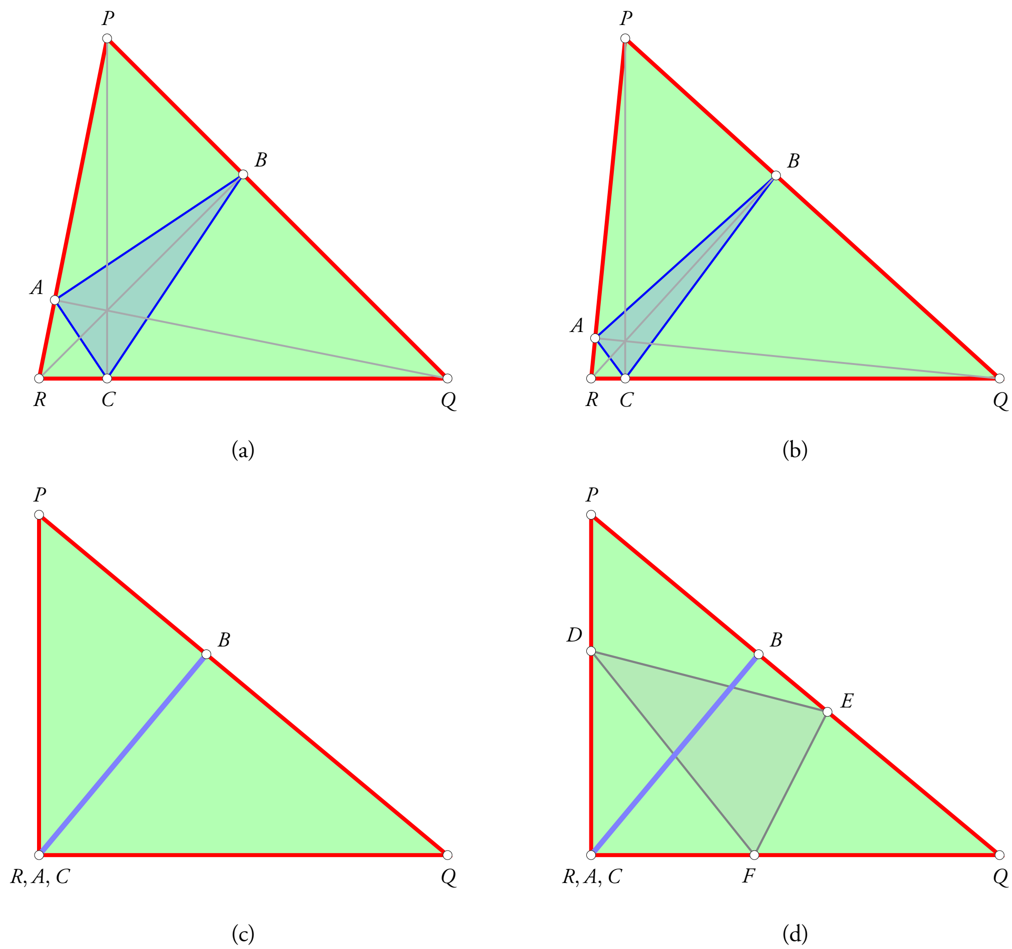


Figure 3

configuration in line QR and again in line PR. The effect is shown in Figure 4; the two mappings take E to points E_1 and E_2 respectively.

We now note the following:

- $E_1F = EF$ and $E_2D = ED$, by the very nature of the reflection operation; hence the perimeter of $\triangle DEF$ is equal to the length of the path E_1FDE_2.
- $\measuredangle ERE_1 = 2\measuredangle ERQ$ and $\measuredangle ERE_2 = 2\measuredangle ERP$, so $\measuredangle E_1RE_2 = 2\measuredangle PRQ = 180°$. That is, points E_1, R, E_2 lie in a straight line.
- The length of path E_1FDE_2 is greater than or equal to the length of segment E_1E_2, i.e., greater than or equal to $2 \times$ the length of segment RE. (This follows from several usages of the result that any two sides of triangle are together greater that the third side.) Hence: perimeter of $\triangle DEF \geq 2 \times$ the length of segment RE.
- The length of segment RE is greater than or equal to the length of segment RB (because RB is perpendicular to PQ).
- Hence: perimeter of $\triangle DEF \geq 2 \times$ the length of segment RB.

The stated claim therefore follows: the optimal inscribed triangle ('optimal' in the sense of having the least possible perimeter) is the degenerate triangle consisting of the segment RB traced twice over. □

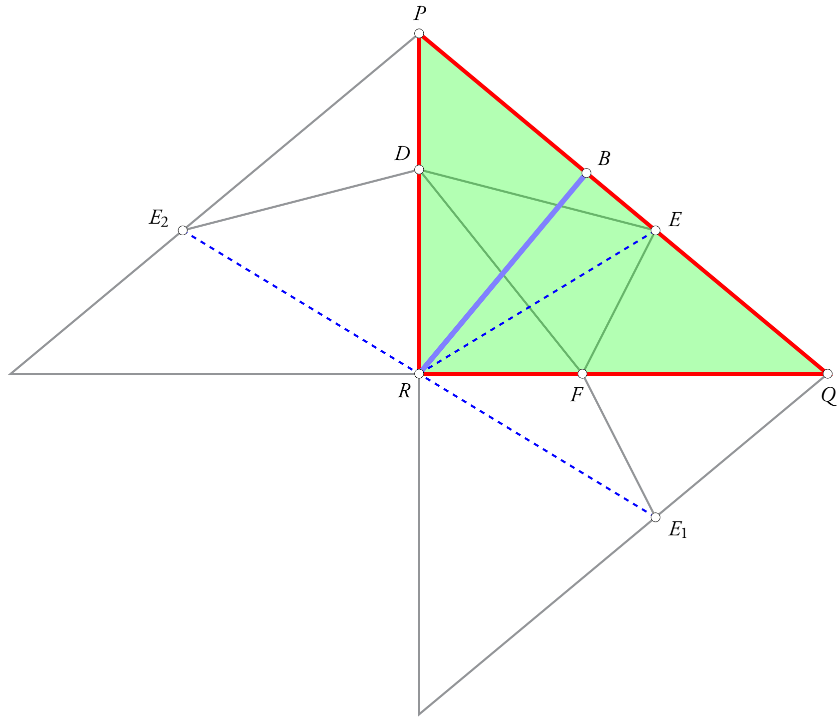
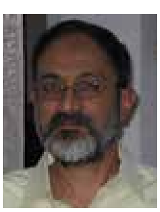


Figure 4

If we continue to increase the size of $\measuredangle PRQ$, then we get a triangle which is obtuse-angled at vertex R. What is the optimal inscribed triangle in this case? It turns out that we cannot do better than opting for the degenerate triangle which consists of the segment RB traced twice over (here, B is the foot of the perpendicular from vertex R to side PQ). We leave the full justification of the statement to you. (Hint: The reflection idea used above will work here as well.) □



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