Problem Corner

Problem Concerning
RATIONAL NUMBERS

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Take any positive rational number and add to it, its reciprocal. For example, starting with 2, we get the sum $2 + 1/2 = 5/2$; starting with 3, we get the sum $3 + 1/3 = 10/3$.

Can we get the sum to be an integer? Clearly we can, in one simple way: by starting with 1, we get the sum $1 + 1/1 = 2$. Observe that the answer is an integer. Is there any other choice of starting number which will make the sum an integer? This prompts the following problem.

Problem: *Find all positive rational numbers with the property that the sum of the number and its reciprocal is an integer.*

Try guessing the answer before reading any further!

We offer two solutions. Some properties of positive integers that we take for granted are the following.

(1) Let p be a prime number and let n be any integer; then: if p divides n^2, then p divides n. Expressed in contrapositive form: *If p does not divide n, then p does not divide n^2.*

(2) *A rational number whose square is an integer is itself an integer.* That is, if x is a rational number such that x^2 is an integer, then x is an integer. Expressed in contrapositive form: *If n is not the square of an integer, then $\sqrt{n}$ is not a rational number.*

Solution I. Let x be a positive rational number such that $x + 1/x = n$ is an integer. We argue as follows.

- Since $x + 1/x \geq 2$ for all $x > 0$, we have $n \geq 2$.

Keywords: *Rational, reciprocal, integer, prime, coprime, divisibility, quadratic, discriminant*

- The relation $x + 1/x = n$ yields the following quadratic equation:
$$x^2 - nx + 1 = 0.$$
Solving it, we get:
$$x = \frac{n \pm \sqrt{n^2 - 4}}{2}.$$
If $n = 2$, the square root term vanishes and we get $x = 1$. This confirms what we already know: that $1 + 1/1$ is an integer.

- If $n \geq 3$, then $2n - 1 \geq 5$, hence $n^2 - (n-1)^2 \geq 5$, which implies that
$$(n-1)^2 < n^2 - 4 < n^2.$$
Therefore $n^2 - 4$ lies strictly between two consecutive perfect squares and cannot be a perfect square itself. It follows that the quantity $\sqrt{n^2 - 4}$ is not an integer. Hence $\sqrt{n^2 - 4}$ is irrational.

- This implies that if $n \geq 3$, then x is an irrational quantity, which is contrary to the given information (namely, that x is a rational number).

- This contradiction shows that if x is rational, then $x + 1/x$ cannot assume any integer value other than 2.

- It follows that 1 is the only positive rational number that has the stated property. □

Solution II. This is a simpler solution. Let the positive rational number r/s be such that the sum of this number and its reciprocal is an integer n. Here we assume that r and s are positive integers that share no factor exceeding 1; i.e., they are coprime. (There is no loss of generality in assuming that r and s are coprime; we would write any fraction in this form.) Let p be a prime divisor of r; then p cannot be a divisor of s. We now have:
$$\frac{r}{s} + \frac{s}{r} = n,$$
$$\therefore\ r^2 + s^2 = nrs,$$
$$\therefore\ s^2 = nrs - r^2.$$

Now a contradiction arises when we check the third equality in terms of divisibility by p; for, p divides r, hence p divides nrs as well as r^2, and so p divides $nrs - r^2$; but p does not divide s^2. These statements clearly contradict each other.

The contradiction shows that there cannot be any such prime divisor p. But this means that $r = 1$, this being the only positive integer with no prime divisor. Swapping the roles of r and s in the above argument, we deduce that $s = 1$ as well. Hence $r = s = 1$, and $r/s = 1$. Thus the only such rational number is 1. □

Questions for further explorations

The main result having been proved, we need to open out the question and ask related questions. For example, we could ask:

Question 1: *With x rational, we cannot get $x + 1/x$ to assume an integer value. But how close can we get to an integer? How close can we get to 3, or 4, or 5 (or any other integer)? What meaningful question can be asked in this regard?*

For example, is the following a reasonable question to ask?

> *Do rational numbers x exist which satisfy the following inequality:*
> $$\left| x + \frac{1}{x} - 3 \right| < 10^{-50}\ ?$$

A possibly more interesting question to explore is the following.

Question 2: *We have already shown that if x is a rational number, then $x + 1/x$ cannot assume an integer value. But what kind of values can it assume? Given an arbitrary rational number a/b, how do we check whether this number lies within the range of the function $f(x) = x + 1/x$?*

It goes without saying that we need a test which can be executed rapidly.

We invite the reader to explore both these questions. Please send in your responses!

Finding the area of a **circle** given the circumference: A **simple** and **quick** method

We know that if, instead of the radius of a circle, the circumference is given, to find the area A enclosed by the circle, we usually find the radius first and then use the formula $A = \pi r^2$. But the following formula gives a quick and easy method to find the area of a circle without finding its radius. Note that we assume that $\pi = \frac{22}{7}$.

Claim: Let C be the circumference of a circle. Then the area A of the circle is given by

$$A = \left(\frac{C}{4}\right)^2 + 33\left(\frac{C}{44}\right)^2.$$

$$\text{Consider RHS} = \left(\frac{C}{4}\right)^2 + 33\left(\frac{C}{44}\right)^2 = \left(\frac{2\pi r}{4}\right)^2 + 33\left(\frac{2\pi r}{44}\right)^2 \qquad (\text{because } C = 2\pi r)$$

$$= \left(\frac{\pi r}{2}\right)^2 + 33\left(\frac{\pi r}{22}\right)^2 = \frac{\pi^2 r^2}{4} + \frac{33\,\pi^2 r^2}{22\times 22}$$

$$= \frac{\pi^2 r^2}{4} + \frac{3\,\pi^2 r^2}{44} = \frac{11\pi^2 r^2 + 3\,\pi^2 r^2}{44} = \frac{14\pi^2 r^2}{44} = \frac{7\,\pi^2 r^2}{22} = \pi r^2 = \text{Area of circle.}$$

Now, we claim the following:

Let C be the circumference of a circle and S be the side of a square such that $C = 4S$. Then the difference between the area of the circle and the area of the square is $33\left(\frac{C}{44}\right)^2$.

Can you prove this using the above formula?

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