ClassRoom

DADS Rule!

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We start with a 10×10 grid numbered sequentially and colour the multiples of 11. As you can see, they occur diagonally and, up to 100, the digits repeat.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 1

As we look at multiples of 11 which are greater than 100, the pattern of repeating digits changes. And to find the more general pattern, look at any other diagonal parallel to the colored one in the 10×10 grid. If we take the diagonal starting with 3 say, we notice the numbers 3, 14, 25, 36, 47, 58, 69, 80. It takes only a moment to notice that there is a pattern in the difference between the units digit and the tens digit. In this diagonal, except for the last difference (which is -8), there is a constant difference of 3. If we take the diagonal beginning with 2, we get the differences as 2 and -9 respectively. We can observe that the two integers we get

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11	22	33	44	55	66	77	88	99	110
121	132	143	154	165	176	187	198	209	220
231	242	253	264	275	286	297	308	319	330
341	352	363	374	385	396	407	418	429	440
451	462	473	484	495	506	517	528	539	550
561	572	583	594	605	616	627	638	649	660
671	682	693	704	715	726	737	748	759	770
781	792	803	814	825	836	847	858	869	880
891	902	913	924	935	946	957	968	979	990

Figure 2

as the digit differences for any diagonal are 11 apart from each other. Now notice the difference between the units and tens digits for the diagonal of multiples of 11; we see that this difference is zero for all multiples of 11 less than 100.

This is a good time to take a look at the 3-digit numbers. We get the multiples of 11 as 110, 121, 132, 143 . . . . Interestingly, the sum of the units digit and the hundreds digit less the tens digit is 0 till we hit 209. There the difference is 11. It seems like it is time to zoom in on the multiples of 11 only. To do this, we advise the use of any spreadsheet (we used Excel) to generate these rows of multiples of 11. If necessary a teacher can always take a printout of these in class. There the students can look into the patterns and color numbers with $(U + H) - T = 11$.

Look at Fig. 2. Here we find an interesting triangle of 209, 308, 407 . . . 902 which yields a difference of alternate digit sums of 11 while every other multiple of 11 gives a zero as the difference of the alternate digit sums.

What about 4 digit multiples of 11? Look at Fig. 3. We observe $(U + H) - (T + Th) = 0$ or 11 or -11. At this point, we decided to name this difference DADS (Difference of Alternate Digit Sums). It is interesting to notice how the triangles with DADS 11 start off as fairly large triangles but then shrink to just one number 7909. Similarly the numbers with DADS -11, appear as triangles with the same orientation initially, but then change orientation and seem to wrap across the ends of the grid.

What if we change the number of columns in the hope of finding some better pattern? We tried using 9 columns, i.e., the 1^{st} row become 11 . . . 99, 2^{nd} row 110 . . . 198, etc. Immediately, we were rewarded by a clearer pattern of triangles. See Fig. 4. The DADS 11 triangles shrink and make room for the DADS -11 ones which increase till they cover almost entire rows.

At this point, we would like to move from narrative mode to posing a few questions which follow the Low Floor High Ceiling Pattern.

3971	3982	3993	4004	4015	4026	4037	4048	4059	4070
4081	4092	4103	4114	4125	4136	4147	4158	4169	4180
4191	4202	4213	4224	4235	4246	4257	4268	4279	4290
4301	4312	4323	4334	4345	4356	4367	4378	4389	4400
4411	4422	4433	4444	4455	4466	4477	4488	4499	4510
4521	4532	4543	4554	4565	4576	4587	4598	4609	4620
4631	4642	4653	4664	4675	4686	4697	4708	4719	4730
4741	4752	4763	4774	4785	4796	4807	4818	4829	4840
4851	4862	4873	4884	4895	4906	4917	4928	4939	4950
4961	4972	4983	4994	5005	5016	5027	5038	5049	5060
5071	5082	5093	5104	5115	5126	5137	5148	5159	5170
5181	5192	5203	5214	5225	5236	5247	5258	5269	5280
5291	5302	5313	5324	5335	5346	5357	5368	5379	5390
5401	5412	5423	5434	5445	5456	5467	5478	5489	5500
5511	5522	5533	5544	5555	5566	5577	5588	5599	5610
5621	5632	5643	5654	5665	5676	5687	5698	5709	5720
5731	5742	5753	5764	5775	5786	5797	5808	5819	5830
5841	5852	5863	5874	5885	5896	5907	5918	5929	5940

Figure 3

A brief recap: an activity is chosen which starts by assigning simple age-appropriate tasks which can be attempted by all the students in the classroom. The complexity of the tasks builds up as the activity proceeds so that students are pushed to their limits as they attempt their work. There is enough work for all, but as the level gets higher, fewer students are able to complete the tasks. The point, however, is that all students are engaged and all of them are able to accomplish at least a part of the whole task.

3080	3091	3102	3113	3124	3135	3146	3157	3168
3179	3190	3201	3212	3223	3234	3245	3256	3267
3278	3289	3300	3311	3322	3333	3344	3355	3366
3377	3388	3399	3410	3421	3432	3443	3454	3465
3476	3487	3498	3509	3520	3531	3542	3553	3564
3575	3586	3597	3608	3619	3630	3641	3652	3663
3674	3685	3696	3707	3718	3729	3740	3751	3762
3773	3784	3795	3806	3817	3828	3839	3850	3861
3872	3883	3894	3905	3916	3927	3938	3949	3960
3971	3982	3993	4004	4015	4026	4037	4048	4059
4070	4081	4092	4103	4114	4125	4136	4147	4158
4169	4180	4191	4202	4213	4224	4235	4246	4257
4268	4279	4290	4301	4312	4323	4334	4345	4356
4367	4378	4389	4400	4411	4422	4433	4444	4455
4466	4477	4488	4499	4510	4521	4532	4543	4554
4565	4576	4587	4598	4609	4620	4631	4642	4653
4664	4675	4686	4697	4708	4719	4730	4741	4752
4763	4774	4785	4796	4807	4818	4829	4840	4851
4862	4873	4884	4895	4906	4917	4928	4939	4950

Figure 4

Summing up our findings so far:

1. All 2 digit multiples of 11 have digits repeated in the tens and units place.
2. For 3 digit multiples of 11, the sum of the units digit and the hundreds digit less the tens digit is either 0 or +11.
3. The DADS (Difference of Alternate Digit Sums) is defined as the difference of the sum of the digits in alternate places).
4. For 4 digit multiples of 11, the DADS was 0, +11 or −11.
5. Numbers which gave a particular DADS value appeared in a triangle, clearly visible in a grid with 9 columns.

Questions for Investigation

1. Are there numbers with DADS equal to 22?
2. Which is the smallest number with DADS equal to 22?
3. Are there numbers with DADS equal to -22? Which is the smallest number with DADS equal to -22?
4. What will be the smallest number with DADS equal to $11n$? How many digits will this number have?
5. Find a general formula for the number of digits of the smallest number with DADS equal to $11n$ for a given value of n.
6. For all multiples of 11, will the DADS be a multiple of 11?

Teacher's Notes:

1. Students may construct numbers of the form

   10101010101010101010101010101010101010101

   to obtain a number with DADS 22. From there it will be a matter of time before they start shortening the number to 20202020202020202020 and seeing if they can get smaller numbers.
2. This is a good chance for students to proceed systematically in an investigation. Following the reasoning in step 1, they obtain the significantly shorter number 909 which has a DADS of 18 (it is not a multiple of 11). To get a DADS of 22, the number will have to be 40909.
3. This is a very interesting variation- following the reasoning in the steps above, we see that 409090 is a number with a DADS of -22. Is this the smallest number? Clearly, if there are non-zero digits in the places alternating with the units place, then the digits in the other places will have to be larger so that the difference remains as -22. Students may notice that the smallest number with a negative DADS will have an even number of digits and the smallest number with a positive DADS will have an odd number of digits. A table recording the smallest number with a particular DADS value will help them make this observation.
4. Going forward, we can ask what will be the number of digits of the smallest number with DADS $11n$ for any natural number. The reasoning is exactly the same as before. E.g. the smallest number with 55 as DADS should have six 9s in every alternate place starting with the units digit and then a 1 in the leading digit ($55 \div 9 = 6$ with remainder 1). It will, therefore be the 13 digit number 1090909090909. So the general formula for the smallest number with a DADS of $11n$ will be as follows: if q and r are natural numbers such that $11n \div 9 = q$ with remainder r (i.e., $r < 9$), then, the number will be

   $$r \times 10^{2q} + 9 \times \left(10^{2(q-1)} + 10^{2(q-2)} + \ldots + 1\right).$$

   It will be a number with $2q + 1$ digits. We call this the DADS Rule!
5. **Proof that if N is a multiple of 11, then DADS is also a multiple of 11, and vice versa:**

   Let us take any number N with $(2n + 1)$ digits as $a_0 + 10a_1 + 100a_2 + \ldots + 10^{2n} \times a_{2n}$

   The alternate digit sums are $a_0 + a_2 + \ldots + a_{2n}$ and $a_1 + a_3 + \ldots + a_{2n-1}$

   And therefore DADS for N is $(a_0 + a_2 + \ldots + a_{2n}) - (a_1 + a_3 + \ldots + a_{2-1})$

   Let us consider N $-$ DADS which is

   $a_0 + 10a_1 + 100a_2 + \ldots + 10^{2n} \times a_{2n} - [(a_0 + a_2 + \ldots + a_{2n}) - (a_1 + a_3 + \ldots + a_{2n-1})]$

   $= a_0 + 10a_1 + 100a_2 + \ldots + 10^{2n} \times a_{2n} - (a_0 + a_2 + \ldots + a_{2n}) + (a_1 + a_3 + \ldots + a_{2n-1})$

$= 11a_1 + 99a_2 + 1001a_3 + 9999a_4 + \ldots + (10^{2n-1} + 1)a_{2n-1} + (10^{2n} - 1)a_{2n}$

$= \sum_{k=1}^{n} [(10^{2k-1} + 1)a_{2k-1} + (10^{2k} - 1)a_{2k}]$

Now $10^{2k-1} + 1 = (10 + 1)(10^{2k-2} - 10^{2k-3} + \ldots. + 1) = 11b$ for some natural number b, i.e., $10^{2k-1} + 1$ is divisible by 11. [This step of successive decomposition may be easier for students to understand if we use an example say

$$\begin{aligned}
10^5 + 1 &= 10 \cdot 10^4 + 1 = 11 \cdot 10^4 - 1 \cdot 10^4 + 1 \\
&= 11 \cdot 10^4 - 10 \cdot 10^3 + 1 = 11 \cdot 10^4 - 11 \cdot 10^3 + 1 \cdot 10^3 + 1 \\
&= 11 \cdot 10^4 - 11 \cdot 10^3 + 10 \cdot 10^2 + 1 = 11 \cdot 10^4 - 11 \cdot 10^3 + 11 \cdot 10^2 - 10^2 + 1 \\
&= 11 \cdot 10^4 - 11 \cdot 10^3 + 11 \cdot 10^2 - 10 \cdot 10 + 1 \\
&= 11 \cdot 10^4 - 11 \cdot 10^3 + 11 \cdot 10^2 - 11 \cdot 10 + 1 \cdot 10 + 1 \\
&= 11 \cdot 10^4 - 11 \cdot 10^3 + 11 \cdot 10^2 - 11 \cdot 10 + 11 - 1 + 1 = 11(10^4 - 10^3 + 10^2 - 10 + 1)
\end{aligned}$$

which is a multiple of 11.

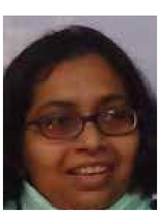
From this step, students may find it easier to generalise. They could also investigate if $10^n + 1$ is a multiple of 11 for all n or only odd n.]

Similarly $10^{2k} - 1 = (10^2)^k - 1 = 100^k - 1 = (100 - 1)(100^{k-1} + \ldots + 1) = 99c$ for some natural number c, $10^{2k} - 1$ is divisible by 99 and hence by 11.

Since N − DADS is divisible by 11, either both N and DADS are divisible by 11 or neither one is; so if DADS is a multiple of 11 so is the original N, and if DADS is not, neither is N.

Conclusion: Mathematical investigations are perfect for Low Floor High Ceiling activities. Here, we have described how a simple pattern can be recognized, investigated, played with and generalized. If your students have enjoyed DADS Rule, do let them try the same strategies with other number patterns; we hope they rule!

And don't forget to share your students' findings with At Right Angles.



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