Drawing A SPIRAL OF SQUARE ROOTS

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"Mathematics possesses a beauty cold and austere, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show." - Bertrand Russell

Pythagoras theorem has been a perennially interesting topic among mathematicians – both young and old. In this investigation, we will create a spiral of sequential square roots using Pythagoras theorem and will explore several related exciting patterns and relationships. Most of these investigation questions can be taken up by students in middle school and high school alike, who have been introduced to the theorem.

Pythagoras Theorem

In any right angle triangle, the square of the hypotenuse, 'c' is equal to the sum of squares of the base and the opposite side 'a' and 'b'. (See Figure 1)

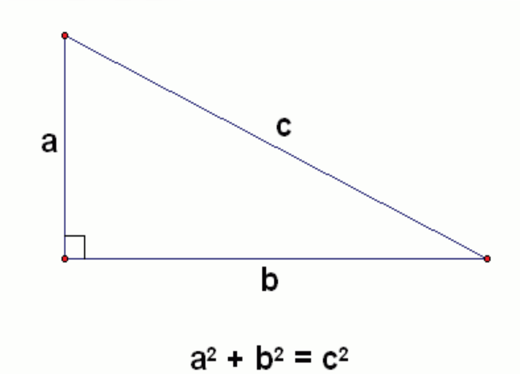

Figure 1

Steps to create a square root spiral

1. Take an A-4 sized paper and draw a line segment AB of unit length in the middle of the paper.
2. At point B, construct a perpendicular line segment of unit length (same as AB), named BC. The hypotenuse AC will hence be of length $\sqrt{\mathbf{2}}$. (See Figure 2)

   Here, $AC^2 = AB^2 + BC^2 = 2$

   So, $AC = \sqrt{\mathbf{2}}$

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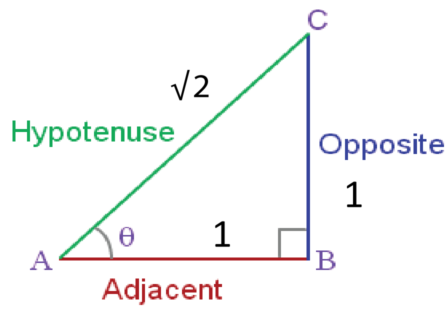

Figure 2

Construct a line at point C perpendicular to segment AC.

Construct line segment CD of unit length (same as AB) on this line.

Then, draw AD to form the hypotenuse with AC as base and CD as the opposite side of the new right-angled triangle. So, $\mathbf{AD} = \sqrt{\mathbf{3}}$. (See Figure 3)

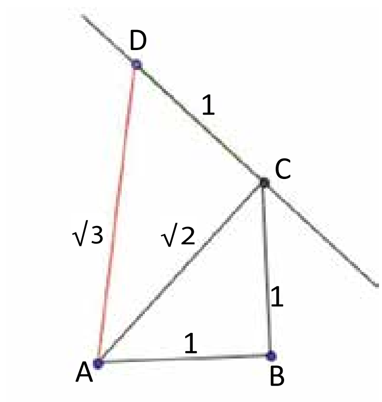

Figure 3

3. Similarly, construct a line segment DE of unit length (same as AB or CD) from point D perpendicular to AD. Join AE to form the hypotenuse of the right-angled triangle with AD as base.

Repeat this process to get more right-angled triangles. The only point to remember is that all the perpendicular opposite sides are of unit length (same as AB).

How many triangles have you drawn? 5, 6,…, 11, 12, 13,…, 16…, n, $n+1$,…?

Investigation questions

1. As you keep drawing, do you notice any shape emerging?
2. At which iteration (n), does this spiral cross the Y-axis?
3. At which iteration (n), does this spiral cross the X-axis?
4. The fan of the spiral seems to be growing longer. Do you notice the same? Is there a way to prove it?
5. How does the angle of triangle at vertex 'A', θ_n, vary with the iterations?
6. What would be the highest possible value of θ_n?
7. What will happen to the spiral if, at each iteration, we vary the length of the opposite side and make it equal to the base? How will the length of the hypotenuse vary now? And how would the angle of the triangle change?

Teachers' Note:

In the following section, the explanations for the above-mentioned investigation questions have been given. Teachers can use it for their reference. Please ensure that students get ample time to explore these questions on their own.

A. As you keep drawing, do you notice any shape emerging?

Yes. You get a shape like a **spiral** as shown in Figure 4.

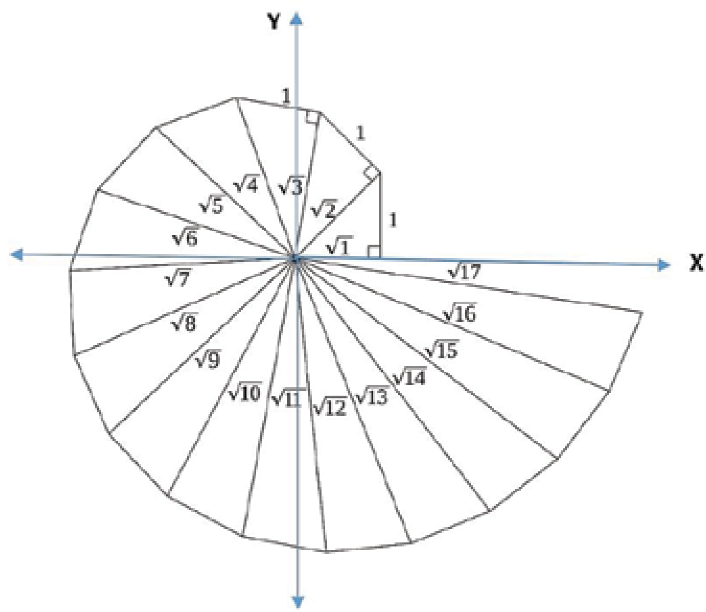

Figure 4

B. At which iteration (n), does this spiral cross the Y-axis?

As seen in Figure 4, the spiral crosses the positive side of the Y-axis at $\boldsymbol{n = 3}$. To see why, note that the sum of the central angles of the spiral is less than 90° at $n = 2$ but greater than 90° at $n = 3$. See Table 2.

When do you think it will cross the negative side of the Y-axis?

Find this out for yourself!

C. At which iteration (n), does this spiral cross the X-axis?

As seen in Figure 4, the spiral crosses the negative side of the X-axis at $\mathbf{n = 6}$. Again, we can confirm this by calculation, the sum of the central angles of the spiral will be less than 180° at $n = 5$ but greater than 180° at $n = 6$.

When do you think it will cross the positive side of the X-axis?

Let us keep drawing. See Figure 5.

D. The fan of the spiral seems to be growing longer. Do you notice the same? Is there a way to prove it?

Remember, when we started, the base and the opposite side of the right angle triangle were of unit length. This made the hypotenuse AC of length $\sqrt{\mathbf{2}}$. (See Figure 2)

Let us record our observations in a tabular format. Enter the length of the base, the opposite side and the hypotenuse. The table is shown below. (See Table 1)

Iteration (n)	**No. of Triangle (n)**	**Base (b)**	**Opposite side (a)**	**Hypotenuse (c)**
1	1	1	1	$\sqrt{2}$
2	2	$\sqrt{2}$	1	$\sqrt{3}$
3	3	$\sqrt{3}$	1	…..
4	4	…...	1	…..
5	5	…...	1	…..
N	n	…...	1	…..

Table 1. Length of base, opposite side and hypotenuse

Looking at the table, it is clear that the opposite side remains of unit length. Only the base and the hypotenuse change. For the n^{th} iteration, what will be the length of the base and the hypotenuse? Can you derive that?

From the tabulated data in **Table 1**, it is clear that for the n^{th}iteration,

Base $= \sqrt{n}$

Hypotenuse $= \sqrt{n+1}$

Thus, for (n+1)$^{\text{th}}$ iteration,

Base $= \sqrt{n+1}$ and

Hypotenuse $= \sqrt{n+2}$

Clearly if $n > 0$,

$\sqrt{n+2} > \sqrt{n+1}$.

It follows that the fan of the spiral (hypotenuse) gets longer with each iteration.

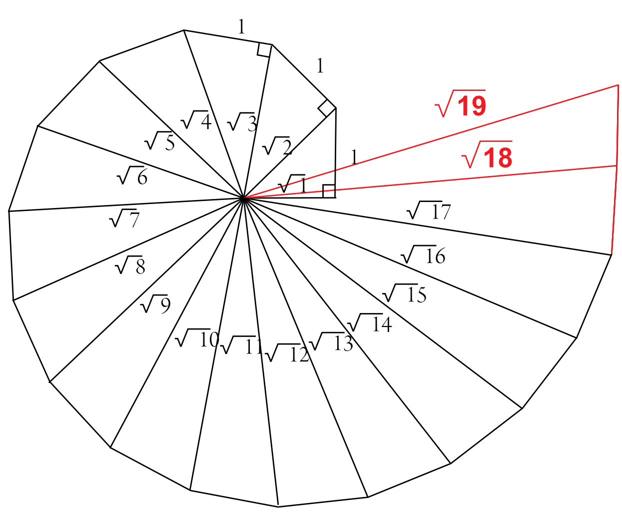

Figure 5

Since we are working with triangles, how can one ignore focusing on the angles?

We have established that with each successive iteration the hypotenuse increases. Can we say the same about the angle, θ_n (the angle of the triangle at the vertex A)? Let us investigate.

E. How does the angle $\boldsymbol{\theta_n}$ of triangle at vertex A, vary with the iterations?

One possible way is to measure the angle $\boldsymbol{\theta_n}$ using a protractor and tabulate the findings as shown in **Table 2.** It would show that the angle steadily decreases.

No. of Triangle (n)	Angle, $\boldsymbol{\theta_n}$
1	45^0
2	......
.....	......

Table 2. Angle of the triangle at each iteration, n

Is there another way to find how the angles vary?

In a right-angled triangle, as the length of base becomes steadily larger, the angle formed by the base and the hypotenuse, $\boldsymbol{\theta_n}$**,** keeps gets steadily smaller. See Figure 6.

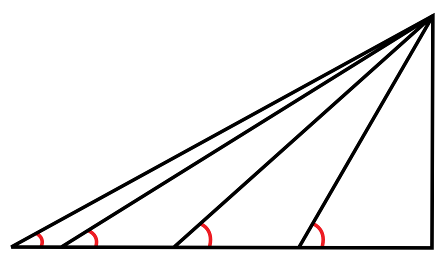

Figure 6

F. What is the largest possible value of $\boldsymbol{\theta_n}$?

Clearly, if the angle decreases with each iteration, the greatest angle is at $n = 1$.

From **Table 2**, at $n = 1$, $\boldsymbol{\theta_1 = 45^0}$**.**

This implies that $\boldsymbol{45^0 \geq \theta_n > 0^0}$**.**

For students in high school, who have studied **trigonometric ratios**, it will be interesting to observe how Sin $\boldsymbol{\theta_n}$ **and** Cos $\boldsymbol{\theta_n}$ vary with each iteration.

We know,

$$\text{Sin } \boldsymbol{\theta_n} = \frac{\text{(Length of opposite side)}}{\text{(Length of hypotenuse)}}$$

From **Table 1,** we know that with each iteration, the length of the opposite side remains equal to 1, but length of the hypotenuse keeps on increasing. Thus, the denominator keeps becoming greater than the numerator. That implies that the value of Sin $\boldsymbol{\theta_n}$ will keep on getting smaller with increase in n.

As shown in **Table 3**,

At $n = 1$, Sin $\boldsymbol{\theta_n} = 1/\sqrt{2}$

i.e. $\boldsymbol{\theta_n = 45^0}$

No. of Triangle (n)	Opposite Side	Base	Hypotenuse	Sin θ_n	Cos θ_n
1	1	1	$\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$
2	1	$\sqrt{2}$	$\sqrt{3}$	$1/\sqrt{3}$	$\sqrt{2}/\sqrt{3}$
3	1	$\sqrt{3}$	…..	…..	……
4	……	……	…..	…..	……
5	……	……	…..	…..	……

Table 3. Length of opposite side, base, hypotenuse and trigonometric ratios at each iteration

G. What will happen to the spiral if, at each iteration, we vary the length of the opposite side and make it equal to the base?

If at each iteration the length of opposite side is made equal to that of the base, it will become an isosceles right-angled triangle and hence, the angle of the triangle will be always equal to 45^0.

What happens to the length of the hypotenuse, now that the length of the base and the opposite side remains the same? Will the hypotenuse grow longer as in the previous case?

Let's draw a table similar to Table 1, capturing how the hypotenuse varies with the iteration. See to Table 4. It shows that the hypotenuse continues to become longer with every iteration.

Iteration (n)	No. of Triangle (n)	Base (b)	Opposite side (a)	Hypotenuse (c)	Angle, θ_n
1	1	1	1	$\sqrt{2}$	45^0
2	2	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{4}$	45^0
3	3	$\sqrt{4}$	$\sqrt{4}$	$\sqrt{8}$	45^0
4	4	$\sqrt{8}$	$\sqrt{8}$	$\sqrt{16}$	45^0
5	5	$\sqrt{16}$	$\sqrt{16}$	$\sqrt{32}$	45^0
6	6	$\sqrt{32}$	$\sqrt{32}$	….	45^0
n	n	$\sqrt{2}\hat{}(n-1)$	$\sqrt{2}\hat{}(n-1)$	$\sqrt{2}\hat{}(n)$	45^0

Table 4. Length of base, opposite side and hypotenuse

In this case, it is also to be noted that the hypotenuse of the triangle at even iterations will overlap with the axes. (Refer *Figure7*)

This can also be established from Table 4 that shows that the angle of the triangle, $\boldsymbol{\theta_n}$, is always equal to 45^0. Hence the sum of vertex angles of triangles of any two consecutive iterations (n and $n+1$) will be 90^0.

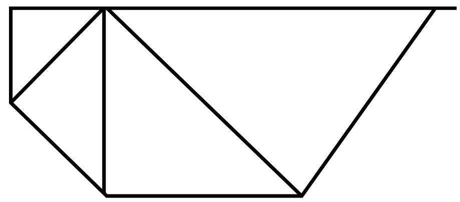

Figure 7

Since at $n = 1$, $\boldsymbol{\theta_n}$=45^0, the hypotenuse of the triangle at even iterations will overlap either with the X-axis or with the Y-axis.

Another interesting observation is that at $n = 8$, sum of angles of triangles at the vertex is equal to 360^0. Thus, the spiral will close.

Will the spiral in the previous case close too? Can you prove it?

It will also be interesting to note the patterns in the length of the hypotenuse and in the angle of the triangles if more variations of this investigation are tried by varying the length of the opposite sides in different manners. For instance,

i. Length of opposite side to be made double the length of the opposite side from previous iteration.
ii. Length of opposite side to be made double the length of the base / adjacent side.
iii. Length of opposite side to be kept equal to the number of iterations, i.e., 1, 2, 3, 4 and so on.

Keep investigating..!

KHUSHBOO is the co-founder and Director at MANTRA Social Services (aka Mantra4Change)- a Bengaluru-based NGO that works to improve the quality of education for the under-served. A management graduate from Tata Institute of Social Sciences (TISS), she has over 8 years of work experience across diverse domains of technology, healthcare, education, product management, business development and stakeholder management. With her newly discovered interest in school education, Khushboo spends her time exploring ways how meaningful learning opportunities can be created in classrooms.