ClassRoom

3 4 5...

STRIKES AGAIN!

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A few days back, I came across the following problem on Dan Meyer's blog **dy/dan** (entry dated October 27, 2016: "How I'm Learning to Step into Math Problems"; I have restated the problem in my own words): *In the figure shown* (Figure 1), *the circle touches the base BC of the square and passes through the two upper vertices, A and D. Find the ratio of the radius of the circle to the side of the square.*

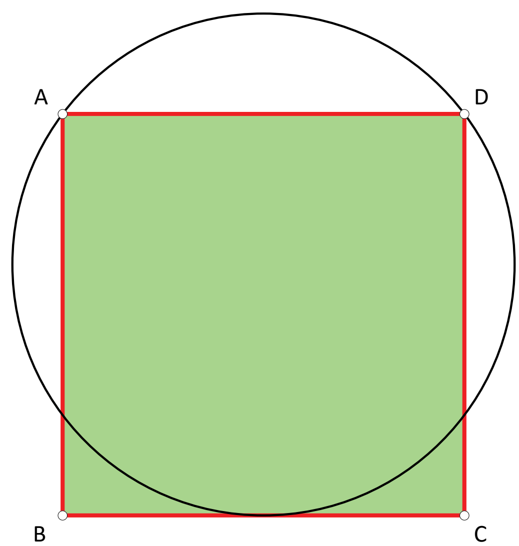

Figure 1

A related problem: *Find a way to construct such a figure*. (That is: given the square, show how to construct the circle; or: given the circle, show how to construct the square.)

Keywords: Pythagoras, problem solving, construction, circle, square

Solution. Denote the radius of the circle by r, and the side of the square by $2a$ (the '2' is only to avoid unnecessary fractions). See Figure 2.

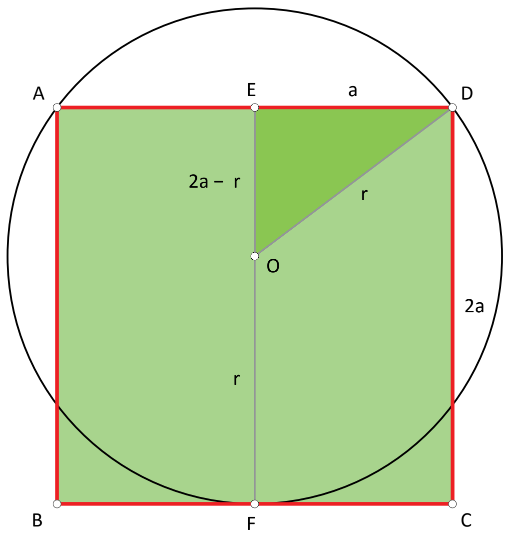

Figure 2

Let O be the centre of the circle, and let EF be the midline of the square, connecting midpoints E and F of AD and BC. (Note that O lies on EF. Why should this be so? In other words: *Given only the information that the circle touches the base BC and passes through the upper two vertices A and D, can we conclude that the midline EF of the square will be an axis of symmetry of the figure?* We leave this question for the reader.) Then we have $OF = r$, $OE = 2a{-}r$, $DE = a$, $OD = r$. Hence from ΔODE we get, using Pythagoras theorem:

$$a^2 + (2a - r)^2 = r^2, \quad \therefore\ 4ar = 5a^2, \quad \therefore\ \frac{a}{r} = \frac{4}{5}.$$

It follows that

$$2a - r : a : r = 3 : 4 : 5,$$

i.e., ΔODE is a 3–4–5 triangle!

Once we have deduced this, the construction procedure becomes clear. All we need to use is the fact that $EO : OF = 3 : 5$ and $FC : r = 4 : 5$.

Remark. A natural extension to the question studied above is obtained by replacing the word ***square*** throughout by ***regular hexagon***. The configuration is depicted in Figure 3. A similar question can now be asked: *Compute the ratio of the radius of the circle to the side of the hexagon.* We leave this question too for the reader to solve. Likewise for more extensions of this kind.

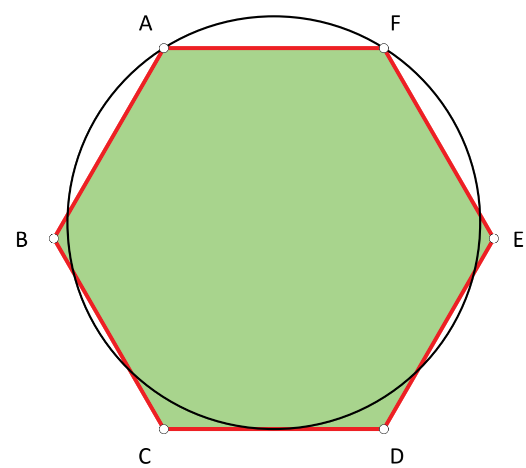

Figure 3

Pedagogical note

These are excellent GeoGebra exercises for students helping them to develop and practise skills of visualisation, logical sequencing, making connections and recalling theory.