# INEQUALITIES in Algebra and Geometry Part 1

ClassRoom

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*This article is the first in a series dealing with inequalities. We shall show that in the world of algebra as well as the worlds of geometry and trigonometry, there are numerous inequalities of interest which can be proved in ways that are easy as well as instructive.*

At first encounter, inequalities tend to unsettle students. Possibly this is because they have gotten used to studying equalities and exact relations. Now all of a sudden they are faced with approximations and inexact relations. Moreover, there is no longer a unique, correct answer to a question! This fact alone can seriously unsettle many students.

Periodically, in our study of inequalities, we will come across implications that are quite counterintuitive; indeed, quite paradoxical. These may be regarded as a bonus.

## Preliminaries: general facts about inequalities

(1) For any two real numbers $a$, $b$, precisely one of the following statements is true:

$$a < b; \qquad a = b; \qquad b < a.$$

This is expressed by saying: *The set $\mathbb{R}$ of real numbers is* ***totally ordered***.

*Remark.* No such statement can be made about the set $\mathbb{C}$ of complex numbers. For example, the numbers 1 and $i$ are non-comparable; no order relation can be placed between them. Note: This is a matter which greatly puzzles students. They ask: *Why cannot we declare that $1 < i$ or that $i < 1$? What would go wrong if we choose to do this?* This is a nontrivial question. We will take it up for study later.

(2) If $x, y$ are real numbers, the following statements are true:

$$x > 0 \text{ and } y > 0 \implies x + y > 0,$$
$$x < 0 \text{ and } y < 0 \implies x + y < 0,$$

and:

$$x > 0 \text{ and } y > 0 \implies xy > 0,$$
$$x > 0 \text{ and } y < 0 \implies xy < 0,$$
$$x < 0 \text{ and } y < 0 \implies xy > 0.$$

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(3) If $x, y, k$ are real numbers, the following statements are true:

$$x < y \quad \text{and} \quad k > 0 \quad \Longrightarrow \quad kx < ky,$$
$$x < y \quad \text{and} \quad k < 0 \quad \Longrightarrow \quad kx > ky.$$

We similarly have the rules which govern the taking of reciprocals:

$$0 < x < y \quad \Longrightarrow \quad 0 < \frac{1}{y} < \frac{1}{x},$$
$$x < y < 0 \quad \Longrightarrow \quad \frac{1}{y} < \frac{1}{x} < 0.$$

For example we have: $0 < 3 < 5$ and $0 < 1/5 < 1/3$; and $-5 < -3 < 0$ and $-1/3 < -1/5 < 0$.

(4) Lastly, we note that the inequality relation has the property of ***transitivity***. This means that if $a, b, c$ are real numbers such that $a < b$ and $b < c$, then we also have $a < c$.

This may seem obvious and innocuous, but in fact the property of transitivity makes its presence felt quite often. Here is how it happens. Say we want to prove the inequality $a < b$ for some two quantities $a, b$; but the nature of the expressions involved makes it difficult to proceed. In such cases an often-used strategy is to look for a third quantity $c$ such that the two inequalities

$$a < c \quad \text{and} \quad c < b$$

are both true and both easily proved. Or we may look for two other quantities $c, d$ such that the separate inequalities

$$a < c, \quad c < d, \quad d < b$$

are all true and all readily proved. Once these steps have been accomplished, the desired inequality follows immediately. The challenge in this case is to find good choices for the quantities $c$ and $d$.

**Remark.** Observe that no rules have been set down governing *subtraction*. Students sometimes imagine that the following is true: if $a < b$ and $c < d$, then $a - c < b - d$ (or some variant of this). But it is very easy to find counterexamples to this claim (it would be an interesting task to set the students: find your own counterexamples). Here is one such: $7 < 8$ and $3 < 5$, but $7 - 3$ is not less than $8 - 5$. Similarly, no rules have been set down governing *division*.

On the other hand, the following *is* true and often made use of: If $a < b$ and $c < d$, then $a - d < b - c$. See if you can justify it for yourself, drawing on the basic facts listed above.

## Theorems and Problems

The most fundamental fact about inequalities is Theorem 1 (below) which may be said to underpin the entire theory of inequalities! We will see this fact at work repeatedly in the next few pages. Two other results of importance are Theorems 2 and 3.

**Theorem 1** (Fundamental fact about inequalities)**.** ***The square of any real number is non-negative.*** *That is, if $x$ is any real number, then $x^2 \geq 0$; equality holds in this relation precisely when $x = 0$.*

**Theorem 2.** *Let $a$ be a positive real number. Then:*

- *If $a > 1$, then $a < a^2 < a^3 < a^4 < \cdots$.*
- *If $a < 1$, then $a > a^2 > a^3 > a^4 > \cdots$.*

**Theorem 3.** *Let $a, b$ be positive real numbers with $a < b$. Then:*

$$a^2 < b^2, \quad a^3 < b^3, \quad a^4 < b^4, \quad a^5 < b^5, \quad \ldots,$$

*and*

$$\frac{1}{a} > \frac{1}{b}, \quad \frac{1}{a^2} > \frac{1}{b^2}, \quad \frac{1}{a^3} > \frac{1}{b^3}, \quad \frac{1}{a^4} > \frac{1}{b^4}, \quad \ldots.$$

We will leave it to you to prove Theorems 2 and 3, by drawing upon facts stated earlier.

In connection with Theorem 2, the following additional statement can be made.

**Theorem 4.** *Let $a$ be a positive real number. Then:*

- *If $a > 1$, then the sequence $1, a, a^2, a^3, a^4, \ldots$ diverges to infinity.*
- *If $a < 1$, then the sequence $1, a, a^2, a^3, a^4, \ldots$ converges to $0$.*

For now, we will not have occasion to use this result.

## Isoperimetric property of the square

As an example of how problems in this topic are handled, we study an extremely well-known problem with a geometrical flavour. Specifically, we prove the following:

**Theorem 5.** *Among all rectangles sharing the same perimeter, the square has the largest area. Among all rectangles sharing the same area, the square has the least perimeter.*

**Proof.** Consider a rectangle with sides $a$ and $b$. Let $p$ be its perimeter, and $k$ its area. Then we have:

$$p = 2(a+b), \quad k = ab.$$

Hence $\dfrac{p^2}{4} = a^2 + 2ab + b^2$, which yields:

$$\frac{p^2}{4} - 4k = a^2 - 2ab + b^2,$$

$$\text{i.e.,} \quad \frac{p^2}{4} - 4k = (a-b)^2.$$

The desired results can be deduced from the last line. Thus we have:

$$4k = \frac{p^2}{4} - (a-b)^2 \le \frac{p^2}{4},$$

and equality holds precisely when $a = b$; hence if $p$ is fixed, then $k$ assumes its largest value when $a = b$, i.e., when the figure is a square. Also:

$$\frac{p^2}{4} = 4k + (a-b)^2 \ge 4k,$$

and equality holds precisely when $a = b$; hence if $k$ is fixed, then $p$ assumes its least value when $a = b$, i.e., when the figure is a square. $\square$

Observe the effective use made of the fact that the square of any real number is non-negative.

## Problems

We now pose a few problems for the reader to work on and if possible solve fully. They can all be solved using the principles listed above. Later in this article we shall present our solutions for your study.

(1) Which is larger: $2^{1/2}$ or 1.5?

(2) Is the square root of 2 closer to 1.4 or to 1.5? (Base your answer to this question on elementary mathematical arguments, and *not* on your existing knowledge of the value of $\sqrt{2}$.)

(3) Is the square root of 3 closer to 1.7 or to 1.75?

(4) Which is larger: $2^{1/2}$ or $3^{1/3}$ (also written as $\sqrt{2}$ and $\sqrt[3]{3}$)?

(5) Which is the largest among the following quantities:

$$\sqrt{1}+\sqrt{19}, \quad \sqrt{2}+\sqrt{18}, \quad \sqrt{3}+\sqrt{17}, \quad \ldots, \quad \sqrt{9}+\sqrt{11}, \quad \sqrt{10}+\sqrt{10}\,?$$

## Solutions

**(1) Which is larger, $2^{1/2}$ or 1.5?**

We use the following: if $a, b > 0$, then $a > b \iff a^2 > b^2$. The squares of the two given numbers are 2 and 2.25 respectively, and 2.25 is obviously the larger quantity. Hence $1.5 > \sqrt{2}$.

**(2) Is the square root of 2 closer to 1.4 or to 1.5?**

We do away with the decimals and ask: which is larger, $a$ or $b$, where

$$a = \left|10\sqrt{2} - 14\right|, \quad b = \left|10\sqrt{2} - 15\right| \,?$$

We have:

$$a^2 = 396 - 280\sqrt{2}, \quad b^2 = 425 - 300\sqrt{2}.$$

Hence:

$$b^2 - a^2 = 29 - 20\sqrt{2}.$$

We need to check whether $b^2 - a^2$ is positive or negative. Since

$$29^2 = 841, \quad \left(20\sqrt{2}\right)^2 = 800,$$

and $841 > 800$, it follows that $b^2 > a^2$, and hence that $b > a$. Therefore the square root of 2 is closer to 1.4 than to 1.5.

**(3) Is the square root of 3 closer to 1.7 or to 1.75?**

We do away with the decimals and ask: which is larger, $a$ or $b$, where

$$a = \left|20\sqrt{3} - 34\right|, \quad b = \left|20\sqrt{3} - 35\right| \,?$$

We have:

$$a^2 = 2356 - 1360\sqrt{3}, \quad b^2 = 2425 - 1400\sqrt{3}.$$

Hence:

$$b^2 - a^2 = 69 - 40\sqrt{3}.$$

We need to check whether $b^2 - a^2$ is positive or negative. Since

$$69^2 = 4761, \quad \left(40\sqrt{3}\right)^2 = 4800,$$

and $4800 > 4761$, it follows that $b^2 < a^2$, and hence that $b < a$. Therefore the square root of 3 is closer to 1.75 than to 1.7.

**(4) Which is larger, $2^{1/2}$ or $3^{1/3}$?**

Let $a = 2^{1/2}$ and $b = 3^{1/3}$. It is obviously very easy to compare two irrational quantities when they have the same exponent. For example, we can readily see that $2^{1/2} < 3^{1/2}$ or that $10^{1/3} < 11^{1/3}$. Here the difficulty is caused by the fact that the two exponents (1/2, 1/3 respectively) are different. So we do the obvious thing: transform the quantities so that the exponents are the same. Since the LCM of 2 and 3 is 6, we raise both $a$ and $b$ to the sixth power. We have:

$$a^6 = \left(2^{1/2}\right)^6 = 2^3 = 8,$$

$$b^6 = \left(3^{1/3}\right)^6 = 3^2 = 9.$$

Since $8 < 9$, it follows that $8^{1/6} < 9^{1/6}$, i.e., $2^{1/2} < 3^{1/3}$.

**(5) Which is the largest among the quantities $\sqrt{1}+\sqrt{19}$, $\sqrt{2}+\sqrt{18}$, $\sqrt{3}+\sqrt{17}$, ..., $\sqrt{9}+\sqrt{11}$, $\sqrt{10}+\sqrt{10}$?** We give two solutions which are both worthy of close study.

**Method 1.** The terms are of the form $\sqrt{x} + \sqrt{20-x}$ for $x = 1, 2, 3, \ldots, 10$. The square of this quantity is

$$x + 2\sqrt{x}\,\sqrt{20-x} + (20-x) = 20 + 2\sqrt{x(20-x)}.$$

It should be evident from this expression that it suffices to find the value of $x$ in the set $\{1, 2, 3, \ldots, 9, 10\}$ for which $x(20-x)$ is the largest. This may be done by multiplying out all the quantities; we find that $x(20-x)$ takes its largest value when $x = 10$. A rigorous proof for this is the following:

$$x(20-x) = 20x - x^2 = 100 - \left(x^2 - 20x + 100\right) = 100 - (x-10)^2,$$

and from the final expression we deduce that

$$x(20-x) \le 100,$$

with equality precisely when $x = 10$. Hence for $x \in \{1, 2, 3, \ldots, 10\}$,

$$\sqrt{x} + \sqrt{20-x}$$

takes its largest value when $x = 10$. Therefore the largest among the given quantities is $\sqrt{10} + \sqrt{10} = 2\sqrt{10}$.

**Method 2.** We shall make clever use of the following identity, which follows from the identity $a^2 - b^2 = (a-b)(a+b)$: for $a \ne b$,

$$a + b = \frac{a^2 - b^2}{a - b}.$$

Comparing $\sqrt{1} + \sqrt{19}$ with $\sqrt{2} + \sqrt{18}$ is the same as comparing $\sqrt{19} - \sqrt{18}$ with $\sqrt{2} - \sqrt{1}$. Now we have, by the identity just quoted:

$$\sqrt{19} - \sqrt{18} = \frac{1}{\sqrt{19} + \sqrt{18}},$$

$$\sqrt{2} - \sqrt{1} = \frac{1}{\sqrt{2} + \sqrt{1}}.$$

Note carefully how use has been made of the fact that $\{1, 2\}$ and $\{18, 19\}$ are pairs of consecutive numbers. Since $\sqrt{19} + \sqrt{18}$ is clearly larger than $\sqrt{2} + \sqrt{1}$, it follows that $\sqrt{19} - \sqrt{18} < \sqrt{2} - \sqrt{1}$, and hence that $\sqrt{1} + \sqrt{19} < \sqrt{2} + \sqrt{18}$.

Next we compare the quantities $\sqrt{2} + \sqrt{18}$ and $\sqrt{3} + \sqrt{17}$. Following exactly the same kind of reasoning, we deduce that $\sqrt{18} - \sqrt{17} < \sqrt{3} - \sqrt{2}$, and therefore that $\sqrt{2} + \sqrt{18} < \sqrt{3} + \sqrt{17}$.

In just the same way, we deduce that:

$$\sqrt{3} + \sqrt{17} < \sqrt{4} + \sqrt{16},$$

$$\sqrt{4} + \sqrt{16} < \sqrt{5} + \sqrt{15},$$

$$\sqrt{5} + \sqrt{15} < \sqrt{6} + \sqrt{14},$$

and so on; and finally:

$$\sqrt{9} + \sqrt{11} < \sqrt{10} + \sqrt{10}.$$

Therefore, the largest of the quantities $\sqrt{1} + \sqrt{19},\ \sqrt{2} + \sqrt{18},\ \sqrt{3} + \sqrt{17},\ \ldots,\ \sqrt{9} + \sqrt{11}$, $\sqrt{10} + \sqrt{10}$ is $\sqrt{10} + \sqrt{10} = 2\sqrt{10}$.

## Problems for you to solve

We close by offering a small list of problems for you to tackle. (No calculators to be used, please!)

(1) Which is larger:

(a) $3^{1/3}$ or $4^{1/4}$?

(b) $4^{1/4}$ or $5^{1/5}$?

(2) Which is larger:

(a) $2^{1/3}$ or $3^{1/4}$ ?

(b) $3^{1/4}$ or $4^{1/5}$ ?

(3) Which is larger: $1.1 + \dfrac{1}{1.1}$ or $1.01 + \dfrac{1}{1.01}$ ?

(4) If $a$, $b$ are nonnegative real numbers with constant sum $s$, what are the least and greatest values taken by $a^2 + b^2$? Express the answers in terms of $s$.

(5) Let $a$, $b$, $c$ be real numbers. Show that:

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$

Under what circumstances does the equality sign hold? In other words, when is it true that $a^2 + b^2 + c^2 = ab + bc + ca$?

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