An example of constructive defining:

From a GOLDEN RECTANGLE to GOLDEN QUADRILATERALS and Beyond

Part 1

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There appears to be a persistent belief in mathematical textbooks and mathematics teaching that good practice (mostly; see footnote[1]) involves first providing students with a concise definition of a concept before examples of the concept and its properties are further explored (mostly deductively, but sometimes experimentally as well). Typically, a definition is first provided as follows:

- *Parallelogram*: A parallelogram is a quadrilateral with half turn symmetry. (Please see endnotes for some comments on this definition.)
- The number $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828\ldots$
- *Function*: A function f from a set A to a set B is a relation from A to B that satisfies the following conditions:
  (1) for each element a in A, there is an element b in B such that $<a, b>$ is in the relation;
  (2) if $<a, b>$ and $<a, c>$ are in the relation, then $b = c$.

[1] It is not being claimed here that all textbooks and teaching practices follow the approach outlined here as there are some school textbooks such as Serra (2008) that seriously attempt to actively involve students in defining and classifying triangles and quadrilaterals themselves. Also in most introductory calculus courses nowadays, for example, some graphical and numerical approaches are used before introducing a formal limit definition of differentiation as a tangent to the curve of a function or for determining its instantaneous rate of change at a particular point.

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Following such given definitions, students are usually next provided with examples and non-examples of the defined concept to 'elucidate' the definition. The problem with this overwhelmingly popular approach is that it creates the misconception that mathematics always starts with definitions, and hides from students that a particular concept can often be defined in many different equivalent ways. Moreover, students are given no idea where the definition came from and on what grounds this particular definition was chosen. By providing students with a ready-made definition, they are also denied the opportunity to engage in the process of mathematical defining themselves, and hence it unfortunately portrays to them an image of mathematics as an 'absolutist' science (Ernest, 1991).

In general, there are essentially two different ways of defining mathematical concepts, namely, *descriptive* (a posteriori) and *constructive* (a priori) defining. Descriptive definitions systematize already existing knowledge, whereas constructive definitions produce new knowledge (Freudenthal, 1973).

The purpose of this article is to heuristically illustrate the process of constructive defining in relation to a recent exploration by myself of the concept of a 'golden rectangle' and its extension to a 'golden rhombus', 'golden parallelogram', 'golden trapezium', 'golden kite', etc. Though these examples are mathematically elementary, it is hoped that their discussion will illuminate the deeper process of constructive defining.

Constructively Defining a 'Golden Rhombus'

> "... [*The*] *algorithmically constructive and creative definition ... models new objects out of familiar ones.*"
>
> – Hans Freudenthal (1973: 458).

Constructive (a priori) defining takes place when a given definition of a concept is changed through the exclusion, generalization, specialization, replacement or addition of properties to the definition, so that a new concept is constructed in the process.

Since there is an interesting *side-angle* duality between a rectangle (all angles equal) and a rhombus (all sides equal) (see De Villiers, 2009:55), I was recently considering how to define the concept of a 'golden rhombus'. Starting from the well-known definition of a golden rectangle as a rectangle which has its adjacent *sides* in the ratio of the golden ratio $\phi = 1.618\ldots$, I first considered the following analogous option in terms of the *angles* of the rhombus (Please see endnotes for the definition of the golden ratio):

A golden rhombus is a rhombus with adjacent *angles* in the ratio of ϕ.

Assuming the acute angle of the rhombus as x, this definition implies that:

$$\frac{180^\circ - x}{x} = \phi, \therefore x = \frac{180^\circ}{1+\phi} \approx 68.75^\circ.$$

An accurate construction of a 'golden rhombus' fulfilling this angle condition is shown in Figure 1.

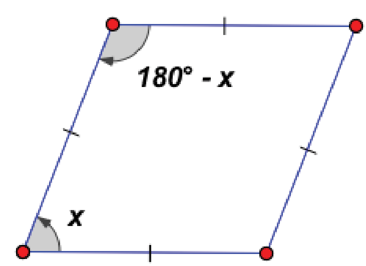

Figure 1. Golden rhombus with angles in ratio phi

Though this particular rhombus looks reasonably visually appealing, I wondered how else one might reasonably obtain or define the concept of a golden rhombus. Since a rectangle is cyclic and a rhombus has an inscribed circle, I hit upon the idea of starting with a golden rectangle *EFGH* (with $\frac{EH}{EF} = \phi$) and its circumcircle, and then constructing the rhombus *ABCD* with sides tangent to the circumcircle at the vertices of the rectangle. (Note that it follows directly from the symmetry of the rectangle *EFGH* that *ABCD* is a rhombus). Much to my surprised delight, I now found through accurate construction and measurement with dynamic geometry software as shown in Figure 2 that though the angles were no longer in the ratio phi, the diagonals for this rhombus now were!

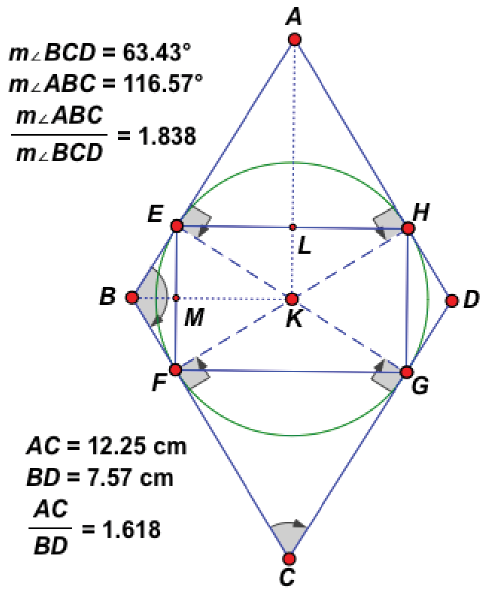

Figure 2. Golden rhombus with diagonals in ratio phi

It is not difficult to explain why (prove that) the diagonals of rhombus *ABCD* are in the ratio ϕ. Clearly triangles *ABK* and *KEM* are similar, from which follows that $\frac{AK}{BK} = \frac{KM}{EM}$. But $KM = LE$; so $\frac{AK}{BK} = \frac{LE}{EM}$. But these lengths (AK, BK); (LE, EM) are respectively half the lengths of the diagonals of the rhombus and the sides of the rectangle; hence the result follows from the property of the golden rectangle *ABCD*.

The size of the angles of the golden rhombus in Figure 2 can easily be determined using trigonometry, and the task is left to the reader. Another interesting property of both the golden rectangle and golden rhombus in this configuration is that $\tan \angle EKF = \tan \angle BCD = 2$. One way of easily establishing this is by applying the double angle tan formula, but this is also left as an exercise to the reader to verify.

Since definitions in mathematics are to some extent arbitrary, and there is no psychological reason to prefer the one to the other from a visual, aesthetic point of view (footnote[2]), we could therefore choose either one of the aforementioned possibilities as our definition. However, it seems that a better argument can be made for the second definition of a 'golden rhombus', since it shows a nice, direct connection with the golden rectangle. Also note that the second definition can be stated in either of the following equivalent forms: 1) a quadrilateral with sides constructed tangential to the circumcircle, and at the vertices, of a golden rectangle as illustrated in Figure 2; or more simply as 2) a rhombus with diagonals in the ratio of ϕ (footnote[3]).

The case for the second definition is further strengthened by the nice duality illustrated between the golden rectangle and golden rhombus in Figure 3, which shows their respective midpoint quadrilaterals (generally called '*Varignon parallelograms*'). Since the diagonals of the golden rectangle are equal, it follows that its corresponding Varignon parallelogram is a rhombus, but since its diagonals are equal to the sides of the golden rectangle, they are also in the golden ratio, and therefore the rhombus is a golden rhombus. Similarly, it follows that the Varignon parallelogram of the golden rhombus is a golden rectangle.

Constructively Defining a 'Golden Parallelogram'

Since the shape of a parallelogram with sides in the ratio of phi is variable, it seemed natural from the aforementioned to define a 'golden parallelogram' as a parallelogram *ABCD* with its sides and diagonals in the ratio phi, e.g., $\frac{AD}{AB} = \frac{BD}{AC} = \phi$ as shown in Figure 4. Experimentally dragging a dynamically constructed general parallelogram until its sides and diagonals were approximately in the golden ratio gave a measurement for $\angle ABC$ of approximately $60°$.

[2]It is often claimed that there is some inherent aesthetic preference to the golden ratio in art, architecture and nature. However, several recent psychological studies on peoples' preferred choices from a selection of differently shaped rectangles, triangles, etc., do not show any clear preference for the golden ratio over other ratios (e.g., see Grossman et al, 2009; Stieger & Swami, 2015). Such a finding is hardly surprising since it seems very unlikely that one could easily visually distinguish between a rectangle with sides in the golden ratio 1.618, or say with sides in the ratio of 1.6, 1.55 or 1.65, or even from those with sides in the ratio 1.5 or 1.7.

[3]A later search on the internet revealed that on https://en.wikipedia.org/wiki/Golden_rhombus, a golden rhombus is indeed defined in this way in terms of the ratio of its diagonals and not in terms of the ratio of its angles. A further case for the preferred choice of this definition can be also made from the viewpoint that several polyhedra have as their faces, rhombi with their diagonals in the golden ratio.

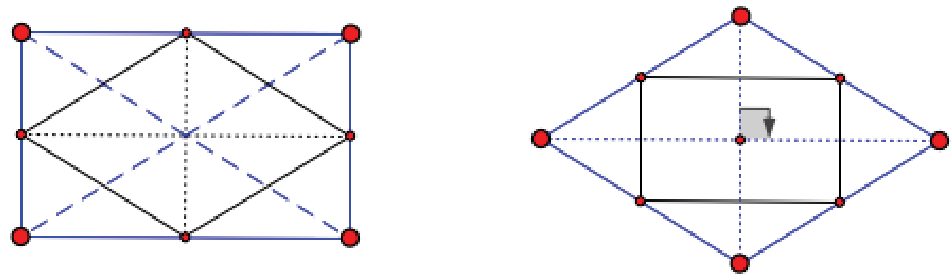

Figure 3. The Varignon parallelograms of a golden rectangle and golden rhombus

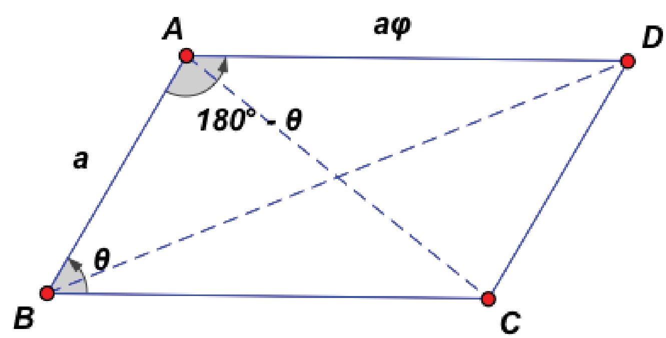

Figure 4. Golden parallelogram with sides and diagonals in the golden ratio

To prove this conjecture was not hard. Assuming $a = 1$ in Figure 4, it follows from the cosine rule that:

$$AC^2 = 1^2 + \phi^2 - 2\phi\cos\theta,$$

$$BD^2 = 1^2 + \phi^2 + 2\phi\cos\theta.$$

But since $\frac{BD}{AC} = \phi$ is given, it follows that:

$$\frac{1^2 + \phi^2 + 2\phi\cos\theta}{1^2 + \phi^2 - 2\phi\cos\theta} = \phi^2.$$

Solving this equation for $\cos\theta$ and substituting the value of ϕ gives:

$$\cos\theta = \frac{\phi^4 - 1}{2(\phi + \phi^3)} = \frac{1}{2},$$

which yields $\theta = 60°$. So my experimentally found conjecture was indeed true. Accordingly, a golden parallelogram defined as a parallelogram with both its sides and diagonals in the golden ratio has 'neat' angles of 60° and 120°, and it also looks more or less visually pleasing. Equivalently, and more conveniently, we could define the golden parallelogram as a parallelogram with an acute angle of 60° and sides in the golden ratio[4] or as a parallelogram with an acute angle of 60° and diagonals in the golden ratio. That the remaining property follows from these convenient, alternative definitions is left to the interested reader to verify.

An appealing property of this golden parallelogram, consistent with that of a golden rectangle, is shown in the first two diagrams in Figure 5, namely, that respectively cutting off a rhombus at one end, or two equilateral triangles at both ends, produces another golden parallelogram. This is because in each case a parallelogram with an acute angle of 60° is obtained, and letting $a = 1$, we see that it has sides in the ratio $\frac{1}{\phi - 1}$, which is well known to equal ϕ.

In addition, constructing the Varignon parallelogram determined by the midpoints of the sides of any parallelogram as shown by the third diagram in Figure 5, it is easy to see that the sides and diagonals of the Varignon parallelogram will be in the same ratio as those of the parent

[4]Somewhat later I found that Walser (2001, p. 45) had similarly defined a golden parallelogram as a parallelogram with an acute angle of 60° and sides in the golden ratio.

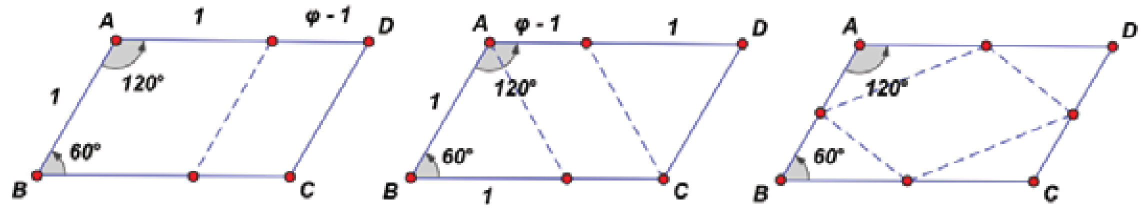

Figure 5. Construction of golden parallelograms by subdivision

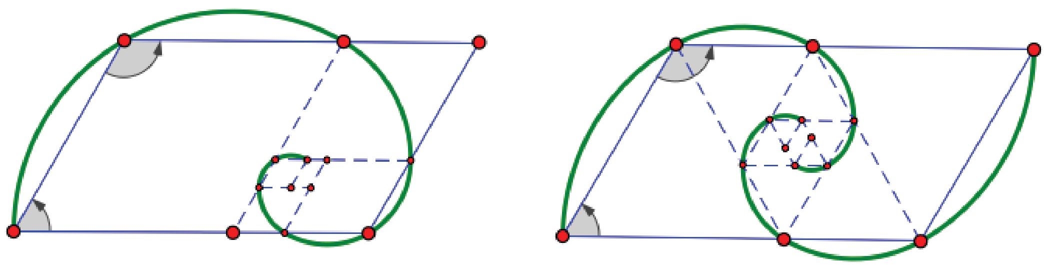

Figure 6. Spirals related to the golden parallelogram

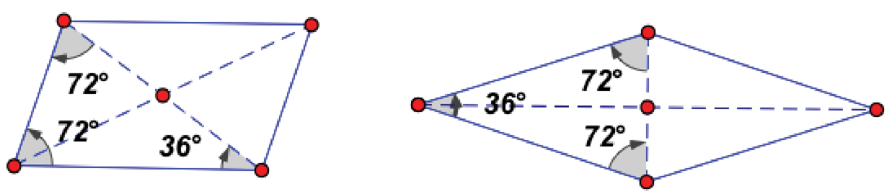

Figure 7. Alternative definitions for golden parallelogram and golden rhombus

parallelogram. Hence, the Varignon parallelogram of a parallelogram will be a golden parallelogram if and only if, the parent parallelogram is a golden parallelogram.

Of further recreational interest is that the subdividing processes of the first two diagrams in Figure 5, can be continued iteratively as shown in Figure 6, just like the golden rectangle, to produce rather pleasant looking spirals.

As was the case with the rhombus, a 'golden parallelogram' can also be constructively defined differently in terms of what is called a 'golden triangle', namely an isosceles triangle with an angle of 36° and two angles of 72° each. (It is left as an exercise to readers to verify that such a triangle has one of its legs to the base in the ratio ϕ). A golden parallelogram can therefore be obtained differently from the aforementioned by a half-turn around the midpoint of one of the legs of the golden triangle to obtain a parallelogram with sides in the ratio ϕ (see footnote[5]) as shown in the first diagram in Figure 7.

Note that using a golden triangle we can also constructively define a golden rhombus in a third way as shown in the second diagram in Figure 7. By simply reflecting a golden triangle around its 'base', we obtain a rhombus with its side to the shorter diagonal in the golden ratio. Though this 'golden rhombus' may appear too flattened out to be visually pleasing, it is of some mathematical interest as it appears in regular pentagons, regular decagons, and in combination with a regular pentagon, can create a tiling of the plane. So this is a case where visual aesthetics of a concept have to be weighed up against its mathematical relevance.

In Part-II of this article, we will explore some possible definitions for golden isosceles trapezia, golden kites, as well as a golden hexagon.

[5] Loeb & Varney (1992, pp. 53-54) define a golden parallelogram as a parallelogram with an acute angle of 72° and its sides in the golden ratio. They then proceed using the cosine rule to determine the diagonals of such a parallelogram to prove that the short diagonal is equal to the longer side of the parallelogram and hence divides it into two golden triangles.

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Endnotes

1. This is not the common textbook definition. (The usual definition is: A parallelogram is a four-sided figure for which both pairs of opposite sides are parallel to each other.) I want to emphasize that concepts can be defined differently and often more powerfully in terms of symmetry. As argued in De Villiers (2011), it is more convenient defining quadrilaterals in terms of symmetry than the standard textbook definitions. Reference: De Villiers, M. (2011). Simply Symmetric. *Mathematics Teaching*, May 2011, p34–36.
2. The Golden Ratio can be defined in different ways. The simplest one is: it is that positive number x for which $x = 1 + 1/x$; equivalently, that positive number x for which $x^2 = x + 1$. The definition implies that $x = (\sqrt{5} + 1)/2$, whose value is approximately 1.618034. A rectangle whose length : width ratio is $x : 1$ is known as a golden rectangle. It has the feature that when we remove the largest possible square from it (a 1 by 1 square), the rectangle that remains is again a golden rectangle.
3. The term Golden Rectangle has by now a standard meaning. However, terms like Golden Rhombus, Golden Parallelogram, Golden Trapezium and Golden Kite have been defined in slightly different ways by different authors.

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