ProblemCroner

Problems for the MIDDLE SCHOOL

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We hope that you enjoyed the reworked Middle Problems and found the Handy Reference Sheet useful. This time we will continue to focus on parity with some nice problems which use the properties given in the sheet.

Problem VI-1-M.1

Can a single coin of value 25 be swapped for 10 coins of value 1, 3 or 5? If yes, describe how this can be done. If no, describe a possible swap by changing any one (and only one) of the numbers given.

Problem VI-1-M.2

A 17 digit number is chosen and its digits are reversed, forming a new number. These two numbers are added together. Show that their sum contains at least one even digit. Is this true for an 18 digit number? If not, give an exception.

Problem VI-1-M.3

Place 4 ones and 5 zeroes around a circle in any order. Now, between any 2 numbers place a zero if the numbers are different and a one if the numbers are the same, and then erase the old numbers. Repeat the same operation on the new numbers. The game continues till all 9 numbers in the circle are the same. How long will this game continue?

These three problems are from *Mathematical Circles* by Dmitri Fomin, Sergey Genkin and Ilia Itenberg.

Problem VI-1-M.4

This is a divisibility game with a set of 0-9 digit cards.

Player 1 chooses any one of the 10 digit cards. Player 2 then chooses any of the remaining cards and places it to the right of the card already there, so that the two digit- number is divisible by 2. The game proceeds in this manner- They take it in turns to choose and place a card to the right of the cards that are already there.

- After two cards have been placed, the two-digit number must be divisible by 2.
- After three cards have been placed, the three-digit number must be divisible by 3.
- After four cards have been placed, the four-digit number must be divisible by 4.

And so on! They keep taking it in turns until one of them gets stuck.

Are there any good strategies to help you to win? What's the longest number you can make that satisfies the rules of the game? Is it possible to use all ten digits to create a ten-digit number?

Is there more than one solution?

http://nrich.maths.org/796

Solutions
Problem VI-1-M.1
The central observation we use to solve the problem is: the sum of an odd number of odd numbers is necessarily odd, and the sum of an even number of odd numbers is necessarily even. Try this with any collection of odd numbers and you will see why it is true. We apply this observation to our problem in which we have to select 10 coins, each of value 1, 3 or 5. Note that the numbers 1, 3, 5 are all odd, whereas 10 is an even number. So the total value of 10 coins is even, no matter how you select the coins. It follows that no collection of 10 such coins can have a value of 25.

The problem becomes possible if 25 is changed to an even number, for example, 30.

Or the number of coins changes to an odd number, for example, 11. If the 11 coins are expressed as the sum of 6,3 and 2 coins then $6 \times 1 + 3 \times 3 + 2 \times 5 = 25$.

Teacher's Note: An easy entry into problems on parity and one that develops logical thinking and communication skills. Students need to defend their reasoning. Opening up the second part of the question gives students a chance to give varied answers all of which could be correct (provided, of course, that they can justify their choices).

Problem VI-1-M.2
Since 17 is an odd number, when it is reversed, the middle (9^{th}) digit does not change position. Since the 9^{th} digit is doubled, it will be an even number (provided there is no carry over from the 8^{th} digit from the right), so the answer will have at least one even digit in this case. If there is a carry over, then let us assume that every digit in the sum is odd. A carry over from the 8^{th} column to the 9^{th}, implies that there is a carry over from the 10^{th} column to the 11^{th} (the digits being added are the same, but in reverse order). Since the sum has given an odd digit and the carry over is 1, obviously the digits in the 11^{th} column are of the same parity. Hence the digits in the 7^{th} column are also of the same parity and there has been a carry over from the 6^{th} to the 7^{th} column. We repeat this argument; there will be a carry over from every even column to every odd column. But it is not possible to have a carry over into the 1^{st} column. So the 1^{st} and the 17^{th} digit should be of opposite parity for the sum in the 1^{st} column to be an odd digit. But then, the numbers in the 17^{th} column will be of opposite parity with a carry over from the 16^{th} column and this will result in an even digit in the 17^{th} column of the sum.

When the number of digits is even, all the digits change position and if the digits which are added are pairs of even and odd numbers then no digit in the sum will be even.

Example: The 18 digit number 123456789012345678. Or, even simpler: 121212121212121212 (with nine repetitions of '12').

Teacher's Note: Again, a chance for reasoning and communication. Plus a chance for students to make judicious use of terms such as *always*, *sometimes* or *never*. For example, it is possible to get an 18 digit number which yields a sum with an even digit. Best of all, students learn to reduce a problem to a simpler version; they should be encouraged to experiment with numbers having a fewer number of digits if they cannot get started on the original version of the problem. Caution: The 3 digit number 637 when reversed and added, gives 1373, a number with no even digit, students can investigate why this happens!

Problem VI-1-M.3

Again, the solution to this problem emerges easily when students take a fewer number of ones and zeros.

A second strategy which helps is to look at the desired outcome- the numbers should be all ones or all zeroes.

If they are all zeroes, then in the previous step, no number should have the same neighbours, or alternate numbers should be the same. And in the step previous to that, the same situation should exist, i.e., no number should have the same neighbours or alternate numbers should be the same. This means that this situation should exist in the original configuration which is not possible, since the number of ones is not equal to the number of zeroes.

If they are all ones, then in the previous step, all the numbers should be the same and that is not possible when both zeroes and ones are used.

Try with 1 one and 1 zero. This works and proceeds to a solution.

With 2 ones and 1 zero, it is impossible to arrange them so that alternate numbers are the same. And this happens whenever the number of zeroes and ones are unequal. So the game goes on forever in this case.

Problem VI-1-M.4

A lot of theory gets internalized when students play games. Divisibility rules can never be learnt by heart, they need to be turned to instinctively and used as quick and effective checks. This problem has terrific opportunities for reasoning; the solution is given on the nrich website, so we will only reveal that if all 10 digits are used, the only possible answer is 3816547290.

Pedagogy: Problem solving strategies that have been imparted with this problem set are:

1. Trying the same problem with a smaller number of cases.
2. Understanding the benefits of trial and error.
3. Learning to generalize and being aware of the pitfalls when this is done.