Problem Corner

ON Problem Posing

PRITHWIJIT DE

Problem-posing and *problem-solving* are central to mathematics. As a student one solves a plethora of problems of varying levels of difficulty to learn the applications of theories taught in the mathematics curriculum. But rarely is one shown how problems are made. The importance of problem-posing is not emphasized as a part of learning mathematics. In this article, we show how new problems may be created from simple mathematical statements at the secondary school level.

We begin with a simple problem.

Problem. Let a, b, c be three positive real numbers. Prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geq\frac{3}{2}. \qquad (1)$$

This is known as *Nesbit's inequality*.

Proof. There are several proofs of this statement. One of them uses the arithmetic mean-harmonic mean (AM-HM) inequality (see Box 1). If we call the algebraic expression on the left hand side P, then by adding 1 to each term we get:

$$\begin{aligned} P+3 &= \left(1+\frac{a}{b+c}\right)+\left(1+\frac{b}{c+a}\right)+\left(1+\frac{c}{a+b}\right) \\ &= (a+b+c)\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right). \end{aligned}$$

Next, by the AM-HM inequality applied to the three numbers $(b+c)/2$, $(c+a)/2$ and $(a+b)/2$, we have

$$\frac{\frac{b+c}{2}+\frac{c+a}{2}+\frac{a+b}{2}}{3}\geq\frac{3}{2\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)},$$

$$\therefore\ \frac{a+b+c}{3}\geq\frac{3}{2\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)},$$

$$\therefore\ (a+b+c)\cdot\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)\geq\frac{9}{2}.$$

Keywords: problem-posing, problem-solving, Nesbit's inequality, arithmetic mean-harmonic mean inequality

Thus $P + 3 \geq \dfrac{9}{2}$ and the desired result follows. It is easy to see that equality holds if and only if $a = b = c$.

The AM-GM-HM inequality

The AM-GM inequality states this: Given any collection of positive numbers, their arithmetic mean is never less than their geometric mean. Moreover, the two means are equal in precisely one situation: the given numbers are all identically equal.

In symbols: Let $a_1, a_2, \ldots, a_n$ be n positive numbers. Their arithmetic mean (AM) and their geometric mean (GM) are defined to be the following:

$$\text{AM} = \frac{a_1 + a_2 + \cdots + a_n}{n},$$

$$\text{GM} = (a_1 a_2 \cdots a_n)^{1/n}.$$

Then we have:

$$\text{AM} \geq \text{GM}.$$

Equality holds in this relation if and only if $a_1 = a_2 = \cdots = a_n$.

The AM-GM inequality may be strengthened to include the harmonic mean (HM). The harmonic mean of n positive numbers $a_1, a_2, \ldots, a_n$ is defined to be:

$$\text{HM} = \frac{n}{1/a_1 + 1/a_2 + \cdots + 1/a_n}.$$

The AM-GM-HM inequality states the following:

$$\text{AM} \geq \text{GM} \geq \text{HM}.$$

Moreover, the equality sign holds if and only if $a_1 = a_2 = \cdots = a_n$.

Numerical example. Consider the four numbers $8, 9, 16, 18$. Then:

$$\text{AM} = \frac{8 + 9 + 16 + 18}{4} = \frac{51}{4} = 12.75,$$

$$\text{GM} = (8 \times 9 \times 16 \times 18)^{1/4} = 12,$$

$$\text{HM} = \frac{4}{1/8 + 1/9 + 1/16 + 1/18} = \frac{192}{17} \approx 11.29.$$

Observe that AM $>$ GM $>$ HM.

The AM-GM-HM inequality is a tremendously useful inequality. It comes of use in a vast number of situations.

More often than not one leaves a problem as soon as a solution is found and does not care to see if there is more to it than meets the eye. But we will explore this problem and create more problems from it by asking more questions and answering them as they come.

It is quite natural to ask whether P is *smaller* than a particular number under the given conditions. To be more precise, we rephrase this as:

Question. Does there exist a positive real number k such that $P < k$ for all positive real numbers a, b, c? Here as earlier,

$$P = \frac{a}{b + c} + \frac{b}{c + a} + \frac{c}{a + b}.$$

Analysis. How does one tackle such a question? One way is to try with small values of k and see if the condition is satisfied. We want a *friendly* value of k greater than $\frac{3}{2}$. Let us try $k = 2$. Now we need to figure out whether $P < 2$ for all positive real numbers a, b and c. The easiest way to settle this is to look at any one term of the expression, say, for instance, the first term $\frac{a}{b+c}$, on the left hand side, and figure out if it can be made large compared to 2. Indeed it can be made equal to 2 by choosing $b = c = 1$ and $a = 6$. Note that this is not the only way. There are numerous choices of a, b and c for which $\frac{a}{b+c} > 2$. Thus we have infinitely many possible choices of a, b and c for which $P > 2$. There is nothing so special about the number 2. If we replace 2 by any positive real number k, then also $\frac{a}{b+c} > k$ for infinitely many positive real numbers a, b and c (one choice is $a = 2k + 2$, $b = c = 1$; it yields $\frac{a}{b+c} = k + 1$). Thus given any positive real number k, we can choose a, b, and c in such a way that $P > k$. This leads us to conclude that there does not exist any positive real number k such that $P < k$ for all positive a, b, c.

Now one may ask under what additional conditions on a, b and c will there exist a positive real number k such that $P < k$ for all positive real numbers a, b and c? What if a, b and c are restricted to assume values over some finite interval, say $(0, 1]$? That is, $0 < a \leq 1, 0 < b \leq 1, 0 < c \leq 1$. Even in this case, P can be made larger than any given positive real number k. Because the fraction $\frac{a}{b+c}$ is unaltered if a, b, and c are replaced by ta, tb, and tc where t is a positive real number such that $ta \leq 1$, $tb \leq 1$ and $tc \leq 1$. But if we demand that a, b, and c satisfy the following:

$$b + c > a, \qquad c + a > b, \qquad a + b > c, \tag{2}$$

then indeed we have $P < 3$. In other words if a, b, and c are the side-lengths of a triangle, then we can find a positive real number $k\,(= 3)$ such that $P < k$. But we can do better. We can make P smaller than 2. How? To see this assume that $c = \max(a, b, c)$. Then observe that

$$\frac{a}{b+c} \leq \frac{a}{a+b}, \qquad \frac{b}{c+a} \leq \frac{b}{a+b}, \qquad \frac{a}{b+c} < 1, \tag{3}$$

which leads to

$$P = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2. \tag{4}$$

Can P ever equal 2? Perhaps the reader may like to ponder over this.

There is another way of proving $P < 2$ by using a very elementary fact about fractions. If x and y are positive real numbers such that $x < y$ then for any positive real number t,

$$\frac{x}{y} < \frac{x+t}{y+t}. \tag{5}$$

The proof is obvious; just cross-multiply and rearrange terms. By virtue of this and the triangle inequality we have

$$\frac{a}{b+c} < \frac{a+a}{a+b+c} = \frac{2a}{a+b+c}. \tag{6}$$

Thus

$$P < \frac{2(a+b+c)}{a+b+c} = 2. \tag{7}$$

Generalization

The next step is to see if we can generalize the results obtained above, to more than 3 variables. The new problem before us is the following.

Problem. Let $n \geq 4$ be a positive integer and let $a_1, a_2, \ldots, a_n$ be positive real numbers. Let

$$Q = \frac{a_1}{a_2 + a_3 + \cdots + a_n} + \frac{a_2}{a_1 + a_3 + \cdots + a_n} + \cdots + \frac{a_n}{a_1 + a_2 + \cdots a_{n-1}}. \tag{8}$$

What is the minimum value of Q?

Note the similarity in form between P and Q. For every term in both P and Q, the numerator and the denominator add up to the same quantity ($a + b + c$ for P and $a_1 + a_2 + \cdots + a_n$ for Q). This suggests using the same approach for Q as we did for P. Let $s = a_1 + a_2 + \cdots + a_n$. Then

$$Q + n = s\left(\frac{1}{s - a_1} + \frac{1}{s - a_2} + \cdots + \frac{1}{s - a_n}\right), \tag{9}$$

and by appealing to the AM-HM inequality we obtain:

$$\frac{\frac{1}{s - a_1} + \frac{1}{s - a_2} + \cdots + \frac{1}{s - a_n}}{n} \geq \frac{n}{ns - s} = \frac{n}{(n-1)s}. \tag{10}$$

Therefore:

$$Q + n \geq \frac{n^2}{n-1}, \tag{11}$$

which yields $Q \geq \frac{n}{n-1}$. We readily observe that $Q = \frac{n}{n-1}$ if $a_1 = a_2 = \cdots = a_n$. Therefore the minimum value of Q is $\frac{n}{n-1}$.

As in the case of P, the maximum value of Q does not exist unless some constraints are placed on $a_1, a_2, \ldots, a_n$. Let us mimic the 3-variable case and demand that

$$s - a_i > a_i \tag{12}$$

for $i = 1, 2, \ldots, n$. Then the conclusion that $Q < n$ is immediate. But what is amazing and perhaps less obvious is that even in this case we can show that $Q < 2$. We use the result on fractions stated earlier. Thus:

$$\frac{a_i}{s - a_i} < \frac{a_i + a_i}{s - a_i + a_i} = \frac{2a_i}{s} \tag{13}$$

and therefore

$$Q < \frac{2(a_1 + a_2 + \cdots + a_n)}{s} = 2. \tag{14}$$

Once again the reader may try to probe if at all Q ever attains the value 2.

Thus we see that starting with a very simple and known result in algebra we could come up with different problems either by way of changing the assumptions or through simple generalizations. Heuristics too played a role in ascertaining whether an algebraic expression would admit an upper bound.

It is not possible to teach a student how to solve each and every problem or how to pose a new one, but perhaps it is possible to plant in him or her the seeds of an inquiry-based approach towards problem-solving or problem-posing.

PRITHWIJIT DE is a member of the Mathematical Olympiad Cell at Homi Bhabha Centre for Science Education (HBCSE), TIFR. He loves to read and write popular articles in mathematics as much as he enjoys mathematical problem solving. His other interests include puzzles, cricket, reading and music. He may be contacted at de.prithwijit@gmail.com.