Problems for the SENIOR SCHOOL

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Problem VI-1-S.1
Let $ABCD$ be a cyclic quadrilateral in which the diagonals AC and BD are perpendicular. Let X be the point of intersection of the diagonals. Let P be the midpoint of BC. Prove that PX is perpendicular to AD.

Problem VI-1-S.2
Let a, b, c be three distinct non-zero real numbers. If a, b, c are in arithmetic progression and b, c, a are in geometric progression, prove that c, a, b are in harmonic progression and find the ratio $a : b : c$.

Problem VI-1-S.3
Prove that the following quantity is not an integer:

$$\frac{2}{3} + \frac{4}{5} + \frac{6}{7} + \cdots + \frac{2016}{2017}.$$

Problem VI-1-S.4
Prove that the product of six consecutive positive integers cannot be a perfect cube.

Problem VI-1-S.5
A square with side a is inscribed in a circle. Find the side of the square inscribed in one of the segments thus obtained.

Solutions to Problems in Issue-V-3 (November 2016)

Solution to problem V-3-S.1

Let ABCD be a convex quadrilateral. Let P, Q, R, S be the midpoints of AB, BC, CD, DA respectively. What kind of a quadrilateral is PQRS? If PQRS is a square, prove that the diagonals of ABCD are perpendicular to each other.

Solution. By the midpoint theorem, PQ and RS are parallel to AC and $PQ = RS = AC/2$. Hence $PQRS$ is a parallelogram. If $PQRS$ is a square, then $PQ \perp QR$. But PQ is parallel to AC and QR is parallel to BD. Thus $AC \perp BD$.

Solution to problem V-3-S.2

Let ABCD be a convex quadrilateral. Let P, Q, R, S be the midpoints of AB, BC, CD, DA respectively. Let U, V be the midpoints of AC, BD, respectively. Prove that lines PR, QS and UV are concurrent.

Solution. Observe that $PQRS$ is a parallelogram. Let PR and QS intersect at X. Note that X is the midpoint of both PR and QS. In triangle ADB, S and V are midpoints of the sides AD and BD respectively. Thus SV is parallel to AB and $SV = AB/2$. In $\triangle ABC$, U and Q are the midpoints of sides AC and BC respectively. Thus UQ is parallel to AB and $UQ = AB/2$. Therefore in quadrilateral $QUSV$ a pair of opposite sides, SV and UQ, are parallel and equal to one another; hence $QUSV$ is a parallelogram. As the diagonals of the parallelogram bisect each other, X must lie on UV. Thus the lines PR, QS and UV are concurrent.

Solution to problem V-3-S.3

There are 12 lamps, initially all OFF, each of which comes with a switch. When a lamp's switch is pressed, its state is reversed, i.e., if it is ON, it will go OFF, and vice-versa. One is allowed to press exactly 5 different switches in each round. What is the minimum number of rounds needed so that all the lamps will be turned ON? (Hong Kong Preliminary Selection Contest 2015)

Solution. Suppose all lamps are turned ON after n rounds. Then we have pressed the switches $5n$ times in all. Note that each lamp must change state an odd number of times. As there are 12 lamps, the total number of times the lamps have changed state is an even number; hence n is even. Clearly $n \neq 2$, since at most $5 \times 2 = 10$ lamps can be turned on in 2 rounds. On the other hand, we can turn on all lamps in 4 rounds as follows:

Round 1: Press switches 1, 2, 3, 4, 5.

Round 2: Press switches 6, 7, 8, 9, 10.

Round 3: Press switches 7, 8, 9, 10, 11.

Round 4: Press switches 7, 8, 9, 10, 12

It follows that the answer is 4.

Solution to problem V-3-S.4

The greatest altitude in a scalene triangle has length 5 units, and the length of another altitude is 2 units. Determine the length of the third altitude, given that it is integer valued.

Solution. Let the triangle be named ABC; let $BC = a$, $CA = b$ and $AB = c$. Since the triangle is scalene, we may assume that $a < b < c$. If h_a, h_b, h_c denote the lengths of the altitudes on sides BC, CA, AB respectively, then $h_a > h_b > h_c$ and $ah_a = bh_b = ch_c = 2\Delta$ where Δ is the area of the triangle. By the condition of the problem, $h_a = 5$. As $a + b > c$ it follows that

$$\frac{1}{h_a} + \frac{1}{h_b} > \frac{1}{h_c}.$$

If $h_b = 2$, then as h_c is an integer the only possibility is $h_c = 1$. But then we get $7/10 > 1$ which is absurd. Thus $h_c = 2$. From the fact that $h_b > h_c = 2$ we get $2 < h_b < 10/3$. Hence $h_b = 3$.

Solution to problem V-3-S.5

If the three-digit number $\overline{ABC}$ is divisible by 27, prove that the three-digit numbers $\overline{BCA}$ and $\overline{CAB}$ are also divisible by 27.

Solution. Since 9 divides $\overline{ABC}$, it divides $A+B+C$. Let $A+B+C = 9m$ for some positive integer m. Now

$$\begin{aligned}\overline{BCA} - \overline{ABC} &= (100B + 10C + A) \\ &\quad - (100A + 10B + C) \\ &= 90B + 9C - 99A \\ &= 9(A+B+C) - 108A + 81B \\ &= 27(-4A + 3B + 3m),\end{aligned}$$

and

$$\begin{aligned}\overline{CAB} - \overline{ABC} &= (100C + 10A + B) \\ &\quad - (100A + 10B + C) \\ &= 99C - 90A - 9B \\ &= -81A + 108C - 9(A+B+C) \\ &= -81A + 108C - 81m \\ &= 27(-3A + 4C - 3m).\end{aligned}$$

Hence $\overline{BCA}$ and $\overline{CAB}$ are divisible by 27.

SOLUTIONS
NUMBER CROSSWORD

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		1 7				2 2		
	3 7	2	4 9		5 3	8	6 1	
7 3	7		8 3	8	0		9 2	0
	10 7	11 2	0		12 8	13 5	4	
		1				0		
	14 3	6	15 6		16 5	0	17 6	
18 6	0		19 7	4	7		20 1	3
	21 1	22 1	1		23 9	24 9	7	
		2				1		

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