Problem Corner

Problems for the SENIOR SCHOOL

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Problem VI-2-S.1

A mathematics teacher wrote the quadratic $x^2 + 10x + 20$ on the board. Then each student either increased by 1 or decreased by 1 either the constant or the linear coefficient. Finally $x^2 + 20x + 10$ appeared. Did a quadratic expression with integer zeros necessarily appear on the board in the process? [From *Polynomials* by Ed Barbeau]

Problem VI-2-S.2

Let $p(t)$ be a monic quadratic polynomial. (The word '**monic**' indicates that the leading coefficient is 1. For example, $x^2 + 10x + 100$ is a monic quadratic; but $3x^2 + 10x + 100$ is not, as its leading coefficient is 3.) Show that, for any integer n, there exists an integer k such that $p(n)p(n+1) = p(k)$. [From *Polynomials* by Ed Barbeau]

Problem VI-2-S.3

Prove that the product of four consecutive positive integers cannot be equal to the product of two consecutive positive integers. [From Round 1, British Mathematical Olympiad, 2011]

Problem VI-2-S.4

Find all integers n for which $n^2 + 20n + 11$ is a perfect square. [From Round 1, British Mathematical Olympiad, 2011]

Problem VI-2-S.5

Find all integers x, y and z such that $x^2 + y^2 + z^2 = 2(yz + 1)$ and $x + y + z = 4018$.

[From Round 1, British Mathematical Olympiad, 2009]

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Solutions to Problems in Issue-VI-1 (March 2017)

Solution to problem VI-1-S.1

Let ABCD be a cyclic quadrilateral in which AC is perpendicular to BD. Let X be the point of intersection of the diagonals. Let P be the midpoint of BC. Prove that PX is perpendicular to AD.

Solution. In triangle BXC, $\measuredangle BXC = 90^\circ$ and P is the midpoint of the hypotenuse BC. Therefore $XP = BP = CP$. Let Q be the point of intersection of PX and AD. Now

$$\measuredangle ADX = \measuredangle ADB = \measuredangle ACB = \measuredangle XCP = \measuredangle PXC = \measuredangle AXQ,$$

and

$$\measuredangle DAX = \measuredangle DAC = \measuredangle DBC = \measuredangle XBP = \measuredangle BXP = \measuredangle DXQ.$$

Therefore in triangle ADX, $\measuredangle ADX = \measuredangle AXQ$ and $\measuredangle DAX = \measuredangle DXQ$. Thus $\measuredangle AQX = \measuredangle DQX = 90^\circ$.

Solution to problem VI-1-S.2

Let a, b, c be three distinct non-zero real numbers. If a, b, c are in arithmetic progression and b, c, a are in geometric progression, prove that c, a, b are in harmonic progression and find the ratio a : b : c.

Solution. The given conditions imply

$$2b = a + c, \quad c^2 = ab.$$

Therefore

$$\frac{2bc}{b+c} = \frac{(a+c)c}{b+c} = \frac{ac+ab}{b+c} = a,$$

so c, a, b are in harmonic progression. Eliminating a we get

$$2b^2 - bc - c^2 = (b-c)(2b+c) = 0.$$

Thus $b = -\dfrac{c}{2}$ and $a = 2b - c = -2c$. Therefore $a : b : c = -2 : -\dfrac{1}{2} : 1 = 4 : 1 : -2$.

Solution to problem VI-1-S.3

Prove that the following quantity is not an integer:

$$\frac{2}{3} + \frac{4}{5} + \frac{6}{7} + \cdots + \frac{2016}{2017}.$$

Solution. Let $S = \dfrac{2}{3} + \dfrac{4}{5} + \dfrac{6}{7} + \cdots + \dfrac{2016}{2017}$. If S is an integer then so is

$$T = 1008 - S = \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2017}.$$

Let $M = 3 \cdot 5 \cdot 7 \cdots 2015$. If T is an integer so is the quantity

$$MT = \frac{M}{3} + \frac{M}{5} + \frac{M}{7} + \cdots + \frac{M}{2017}.$$

Each of the terms except the last term is an integer. Therefore $\frac{M}{2017}$ is an integer. But as 2017 is prime, it does not have any factor in common with M and hence does not divide M. Thus we arrive at a contradiction. Hence S cannot be an integer.

Solution to problem VI-1-S.4

Prove that the product of six consecutive positive integers cannot be a perfect cube.

Solution. Because $6! = 720$ is not a perfect cube, we only need to consider products $t = n(n+1)(n+2)(n+3)(n+4)(n+5)$ with $n \geq 2$. Now

$$t = a^3 + 10a^2 + 24a$$

where $a = n(n+5)$. Because $a \geq 14$, we have:

$$\begin{aligned}(a+3)^3 &= a^3 + 9a^2 + 27a + 27 \\ &= t - (a-9)(a+3) - 3a < t < a^3 + 12a^2 + 48a + 64 \\ &= (a+4)^3.\end{aligned}$$

Hence t cannot be a perfect cube.

Solution to problem VI-1-S.5

A square with side a is inscribed in a circle. Find the side of the square inscribed in one of the segments thus obtained.

Solution. The radius of the circle is $\frac{a}{\sqrt{2}}$. By symmetry it is clear that the sides of the square inscribed in the segment are parallel to the sides of the bigger square. Let x be the side length of the inscribed square. Then:

$$\left(\frac{a}{2} + x\right)^2 + \left(\frac{x}{2}\right)^2 = \frac{a^2}{2}.$$

This reduces to

$$5x^2 + 4ax - a^2 = 0.$$

Thus $x = \frac{a}{5}$.