ClassRoom

Fagnano's Theorem ALTERNATIVE PROOF

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Fagnano's Problem: In 1775, Giovanni Fagnano posed and solved the problem - "For a given acute angled triangle, determine the inscribed triangle of minimum perimeter." Using calculus, Fagnano showed the solution to be the Orthic Triangle – a triangle formed by the feet of the three altitudes. A different proof was given in At Right Angles, Vol. 6, No. 1, March 2017, available at http://azimpremjiuniversity.edu.in/SitePages/resources-ara-march-2017-fagnanos-problem.aspx

Let's see if it can be solved using geometry and a result from Physics called Fermat's principle.

Fermat's principle states that between two points, light always follows the path that takes the shortest time. In case of pure reflection, it is the path of the shortest length too.

Hence, if the given acute triangle is formed using three mirrors PQ, QR, RP and a ray of light is '*perpetually trapped*' between them by getting repeatedly reflected at points A, B and C, the loop CBAC will be the shortest path that reflects a ray from point C back to itself. See Figure 1.

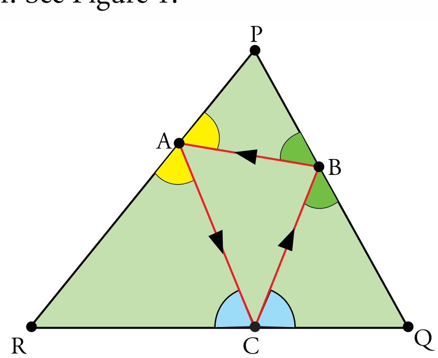

Figure 1. By **Fermat's Principle,** a ray of light perpetually trapped between mirrors PQ, QR and RP gives the shortest path from point C to itself.

Keywords: Geometry, Fagnano's Theorem, Optimization, Minimization, Perimeter

The path of such a repeated reflection can be plotted on the original triangle, its image P′Q′R′ (in mirror PQ) and an image of that image – or its double image - P″Q″R″ (in mirror P'R'). See Figure 2.

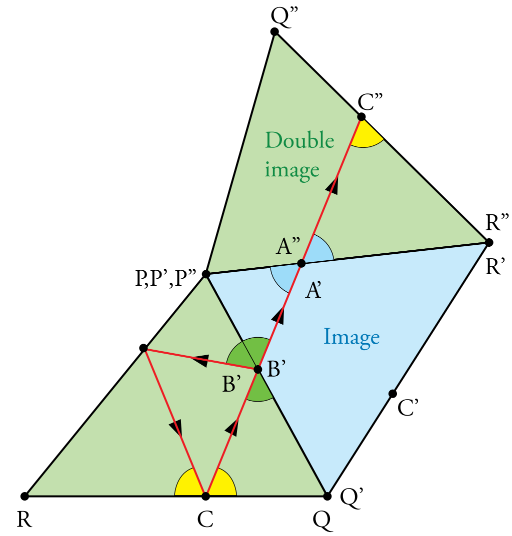

Figure 2. The loop CBAC of repeated reflection of light can be unfolded on the given triangle and its two successive images. It connects point C to its double image C" with a straight line.

Note that the double reflection P″Q″R″ is merely a rotation of the original triangle PQR!

The closed path CBAC then unfolds into a straight line from point C to its double image C″.

The rectilinear nature of the unfolded loop shows that such a loop must be unique and shortest in length as well.

Of course, there would be infinitely many such loops or straight lines – one for every starting point C.

So, which of these loops would be the triangle Fagnano was looking for?

To find out, let's consider an arbitrary point D, its double image D″ and the triangle PDD″ (see Figure 3).

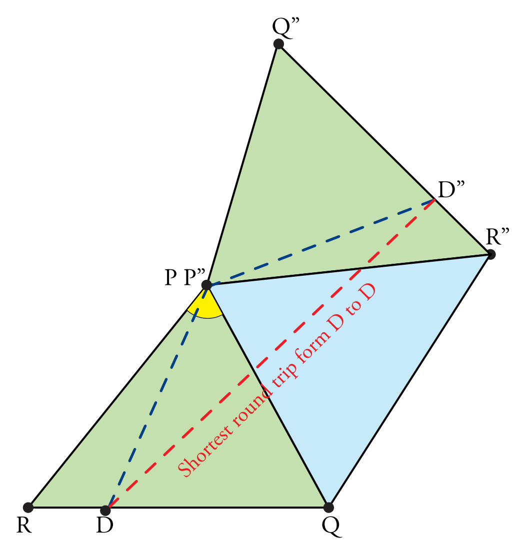

Figure 3. All triangles PDD″ are isosceles with identical vertex angle 2γ and hence similar. For base DD″ to be the shortest, side PD should also be minimized.

Sides PD and PD″ must be identical, since PD″ is simply the double image of side PD.

Thus, triangle PDD″ must be isosceles.

Moreover, since a double reflection really amounts to a simple rotation through twice the angle RPQ = γ, the vertex angle DPD″ for all such isosceles triangles must be the same, viz., 2γ, making them all similar.

The triangle we seek would have the shortest possible base DD″.

Since similar triangles are scaled replicas of each other, the shortest possible base belongs to the smallest of the triangles, which in turn will have the shortest possible sides PD or PD″ too. And this condition is met when PD is the perpendicular dropped from P on to side RQ (shortest distance of a point from a line).

Hence the shortest path is obtained when point D is the foot of the perpendicular. By symmetry, this applies to all the three points A, B and C, making the shortest path the so called Orthic triangle! See Figure 4.

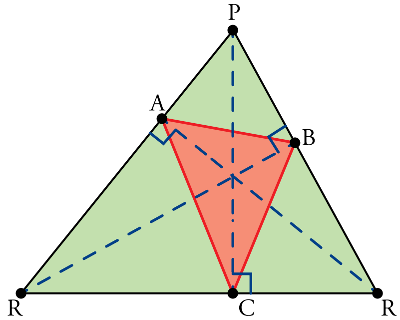

Figure 4. The minimum perimeter inscribed triangle connects the feet of the three altitudes forming an **Orthic Triangle**

Please see the YouTube video at https://youtu.be/5MrNM-VxXd8 for an animation of the above.

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