Low Floor High Ceiling Tasks

If you reason, you can begin with…

APC - AREA, PERIMETER AND CONGRUENCY

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We continue our Low Floor High Ceiling series in which an activity is chosen – it starts by assigning simple age-appropriate tasks which can be attempted by all the students in the classroom. The complexity of the tasks builds up as the activity proceeds so that students are pushed to their limits as they attempt their work. There is enough work for all, but as the level gets higher, fewer students are able to complete the tasks. The point, however, is that all students are engaged and all of them are able to accomplish at least a part of the whole task.

The topic of Perimeter and Area provides rich ground for teachers to examine the truth values of statements and then introduce the crucial 'What-If' which can change a situation around completely. While the topics of area, perimeter and congruency are addressed in class 9 and 10, students will also need to know some trigonometry including the relationship between the sines of supplementary angles, area in terms of the sine of the enclosed angle and the cosine rule to do these tasks. In case these are not familiar to the students, we give them below.

1. $\sin\theta = \sin(180 - \theta)$
2. Area of a triangle $= \frac{1}{2}\, a\, b \sin\theta$, where a, b are the lengths of the sides enclosing the angle θ
3. $\cos\theta = \dfrac{a^2 + b^2 - c^2}{2\, ab}$ in a triangle with sides of length a, b, c where c is the side opposite the angle θ

Keywords: Polygon, area, perimeter, angles, congruency, mensuration, trigonometry, conditional statement, exploration, skill development.

The tasks are in the nature of explorations and give students plenty of opportunity to do ruler and compass constructions or create GeoGebra sketches. In each of the tasks, the underlying question is: Given some equal parameters, are two polygons congruent? In exploring, conjecturing, justifying and proving, students are able to create conditional statements that illustrate the delicate art and science of mathematical reasoning.

Task 1

Consider two identical (congruent) rectangles.

- Do they have the same perimeter?
- Do they have the same area?
- Explain why this is so.

Now consider two rectangles with the same perimeter.

- Do they have the same area?
- Prove your answer.

Consider two rectangles with the same area.

- Do they have the same perimeter?
- Justify your stance.

Finally, consider two rectangles with the same perimeter and the same area.

- Are they identical?
- Prove your answer.

Teacher's Notes: A nice easy start to proof in the mathematics class! Rectangles provide a gentle introduction to mensuration and students are also able to justify their stance in words and in some cases, simply with a numerical example that is an exception to the statement. An 'if and only if' statement can be framed and the teacher can use it as an example for the same.

Task 2

- The congruence of two triangles implies equality of their areas and perimeters. True or False? Justify your stance.
- Find the area and perimeter of the two triangles with sides 20,21,29 and 17,25,28. Are they congruent? Justify.
- Use the above questions and their answers to frame a statement that connects the congruency of two triangles with their area and perimeter.
- If two similar triangles have the same perimeter, will they be congruent?
- If two similar triangles have the same area, will they be congruent?

Teacher's Note: A nice refresher course for mensuration formulae. Students should be facilitated to frame the statement that 'if two triangles are congruent, then their area and perimeter are the same, but if they have the same area and perimeter, they need not be congruent.'

In this task, students get practice in framing a conditional statement. In addition, they are able to deepen their understanding of the concept of 'if and only if' with both examples as well as counter-examples.

Students' understanding of similarity can lead them to conclude that when either the area or the perimeter is the same for similar triangles, the common ratio is 1, implying congruency.

Task 3

Now consider two triangles with equal area and perimeter and having one pair of equal sides. Are they congruent? Prove your statement.

Teacher's Note: This task may need some support from the teacher, but with a little support in writing up the given data in symbolic form and using the appropriate mensuration formulae, students should be able to arrive at a satisfactory conclusion with a rigorous proof. One proof has been given below.

Area, perimeter and one side equal
Let the sides of the triangle be a, b, c and u, v, c. Since the perimeters are equal, let $p = a + b + c = u + v + c$. Since the areas are equal we have (by Heron's formula),

$$p(p-2a)(p-2b)(p-2c) = p(p-2u)(p-2v)(p-2c). \quad (1)$$

Upon simplification and observing that $a + b = u + v$ we obtain

$$ab = uv. \quad (2)$$

Thus we now need to ascertain whether the two equations: $a + b = u + v$ and $ab = uv$ imply $\{a, b\} = \{u, v\}$. Observe that

$$(x-a)(x-b) - (x-u)(x-v) = 0 \quad (3)$$

for all real values of x. By substituting $x = a$ and $x = b$ in turn shows that $\{a, b\} = \{u, v\}$. Hence the triangles are congruent by SSS.

Task 4
Is there any other condition under which triangles with the same area and the same perimeter become congruent?

Teacher's Note: Now that the class has got one condition for two triangles with the same area and the same perimeter to be congruent, experiment with sides and angles to find other such conditions. Here is one of our discoveries.

Area, perimeter and one angle equal
Let the sides of the triangle be a, b, c and u, v, w. Let the common angle be θ and in the first triangle let the sides a and b include θ, and in the second triangle let the sides that include θ be u and v. As the areas are equal we have

$$\frac{1}{2}ab\sin\theta = \frac{1}{2}uv\sin\theta \quad (4)$$

whence $ab = uv$. The cosine rule gives us

$$\cos\theta = \frac{a^2+b^2-c^2}{2ab} = \frac{u^2+v^2-w^2}{2uv} \quad (5)$$

Therefore

$$a^2+b^2-c^2 = u^2+v^2-w^2, \quad (6)$$

and upon adding the equal quantities $2ab$ and $2uv$ to the left hand side and right hand side respectively of the previous equation we obtain

$$(a+b)^2 - c^2 = (u+v)^2 - w^2. \quad (7)$$

Factoring both sides and using the fact that $a + b + c = u + v + w$ leads to

$$a + b - c = u + v - w, \quad (8)$$

whence $c = w$ and we end up as in the previous case (area, perimeter and one side equal). Thus the two triangles are congruent.

Task 5
- If two triangles have the same perimeter and two pairs of equal sides, are they congruent?
- Now, construct two triangles having the same area and with two pairs of sides the same.
- Are they congruent?
- Prove your statement.

Teacher's Note: Do discuss with your students how each statement differs from the previous statements – it's a good exercise in subtle alteration of constraints!

A little bit of thinking is required before students do the construction. We used the theorem that triangles with the same base and between the same parallel lines have the same area. Constructions could be done with ruler and compass or on GeoGebra.

Through D draw a line parallel to AB (see Figure 1). With A as centre and AD as radius draw an arc of a circle. Let this arc intersect the line parallel to AB through D again at E. Join BD. The triangles under consideration are $\triangle ABD$ and $\triangle ABE$.

The construction (by use of a counter-example) will give students an indication that the triangles need not be congruent; we give the justification below but strongly advise you to allow the students to explore and conjecture before going into a rigorous proof.

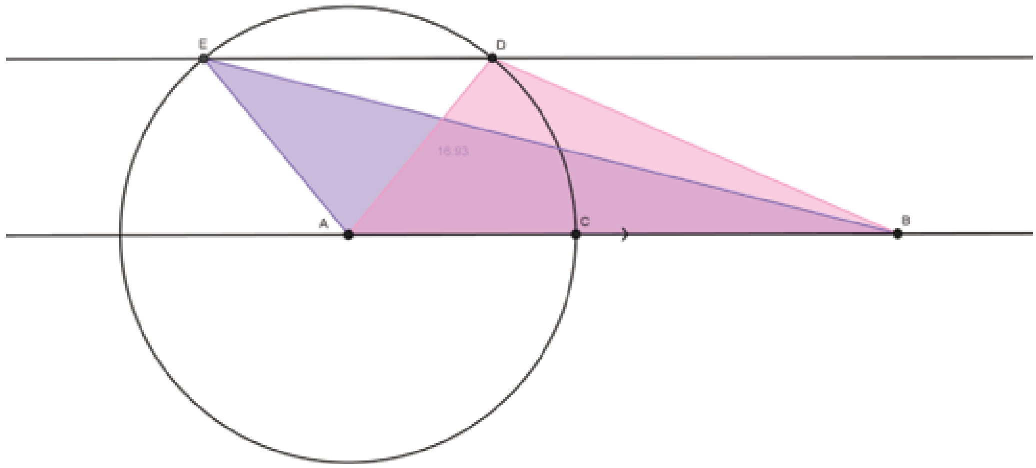

Figure 1. Triangles of the same area with two sides in common

Let the two common sides of the two triangles be $AB = a$ and $AD = AE = b$. If $\triangle$ denotes the common area then the angle included by the common sides, θ_1 in the first triangle and θ_2 in the second triangle, satisfy

$$\sin\theta_1 = \sin\theta_2 = \frac{2\triangle}{ab}. \tag{9}$$

Therefore, either $\theta_1 = \theta_2$ or $\theta_1 = 180^\circ - \theta_2$, and hence in this case we cannot conclude, barring when $\theta = 90^\circ$, that under the given conditions the triangles are congruent. Note that the diagram makes this obvious.

Task 6
Sketch two triangles with the same area, having one side and one angle in common. Are they congruent?

Teacher's Note: Here the challenge for the students is to identify three cases, one in which the common angle has the common side opposite it for both triangles, the second in which the common side is an arm of the common angle for both triangles and a third in which the common angle is opposite the equal side in one triangle and adjacent to the equal side in another.

A proof for the first two cases is given below.

If the common side is one of the sides that include the common angle then the triangles are congruent. To see this, let the common side be a and the common angle θ. In the first triangle let the sides that include θ be a and x, and the same in the second triangle be a and y.

Then equality of area forces

$$\frac{1}{2}ax\sin\theta = \frac{1}{2}ay\sin\theta \tag{10}$$

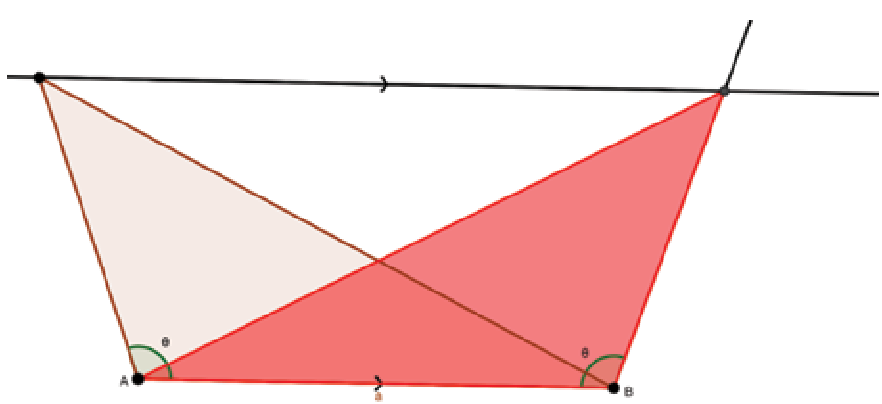

Figure 2. Two triangles of the same area with one common side and one equal angle adjacent to the side

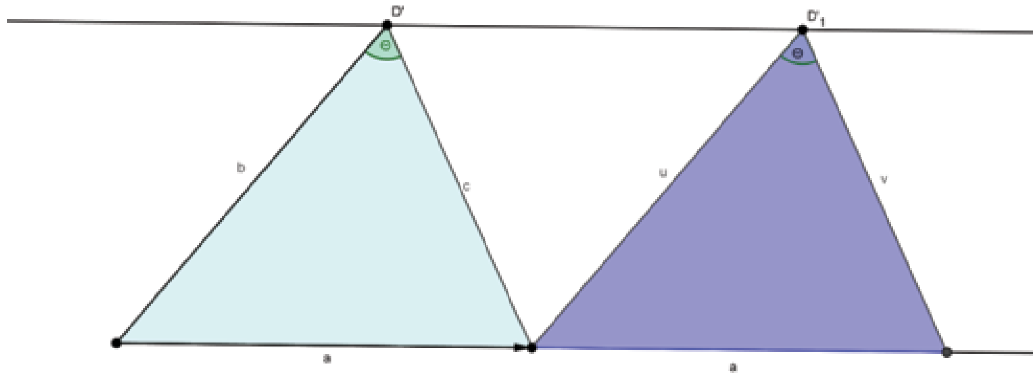

Figure 3. Triangles with the same area having one pair of sides equal and one equal angle opposite the equal sides

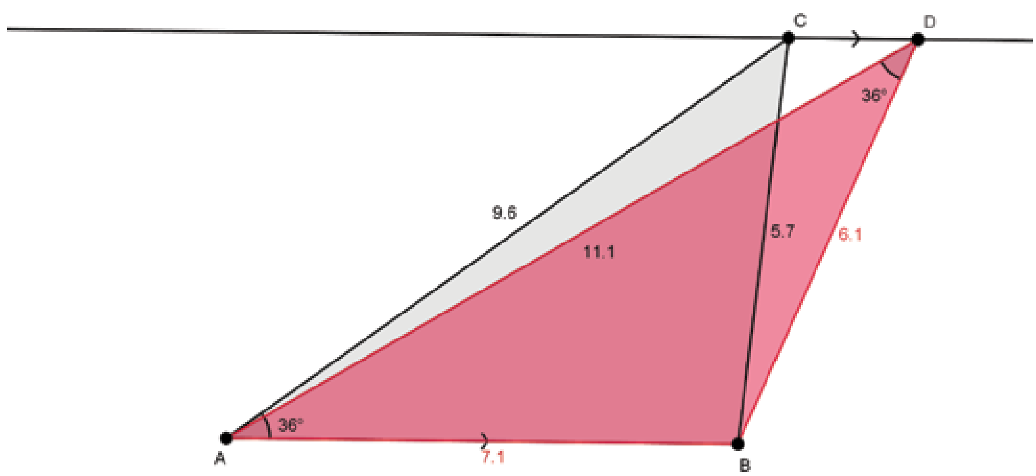

Figure 4. Triangles with the same area, one common side.
One common angle is situated adjacent to the common side in one triangle and opposite the common side in the next

whence $x = y$ and the triangles are congruent by SAS.

What if the common side is situated opposite the common angle? Let b and c be the sides that include θ in the first triangle and let u and v be the corresponding sides in the second triangle.

Then using the equality of areas,

$$bc = uv \tag{11}$$

and

$$\begin{aligned}\frac{b^2+c^2-a^2}{2bc} &= \frac{u^2+v^2-a^2}{2uv}\\ \implies b^2+c^2 &= u^2+v^2\\ \implies (b+c)^2 &= (u+v)^2\\ \implies (b+c) &= (u+v),\end{aligned} \tag{12}$$

and we see that the two triangles have equal perimeter. Since they have equal area, equal perimeter and a side in common, they are congruent (refer to task 2).

Construction of the third situation gives us two triangles which are not congruent. In Figure 4, $\triangle ABC$ and $\triangle ABD$ have the same area, common base and $\angle A = \angle D$. The corresponding heights (from C and D respectively) must be equal. The figure shows that the triangles are not congruent; the exception proves that these conditions do not imply congruency. An interesting tangential investigation is to show that this construction is not possible for all values of $\angle A$; the reader may want to find which values, at leisure.

Task 7
Write up your conclusions from all the tasks above.

Teacher's Note: With so many options considered, the high ceiling in this case, is for students to collate all their learning from the tasks and present it in a compiled form.

Conclusion: This exploration has scope for practising the skills of visualization, logical reasoning, proof and communication, among others. Students may enjoy devising other cases to study, such as replacing equal area with equal perimeter in Task 6. We look forward to hearing from you as you break the class ceiling!

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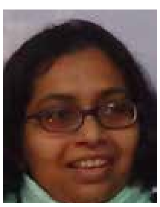

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