Complex

Napoleon's Theorem Part 1

Made Simple

In this article we discuss a gem from Euclidean geometry that was discovered in post-revolution France, a result known as Napoleon's theorem.

Features

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It is rare to come across a result belonging to Euclidean geometry which does not date all the way back to some ancient Greek mathematician (Euclid; Pythagoras; Thales; Archimedes; Apollonius), or some ancient Indian mathematician (Brahmagupta). One such result—Morley's theorem—has been the subject of a three-part series of articles in earlier issues of this magazine. In this note, which will also be in three parts, we discuss another such gem whose discovery goes back to nineteenth century France: a result known as *Napoleon's theorem*. The feature it shares with Morley's theorem is the unexpected occurrence of an equilateral triangle within a given triangle. However, it is far easier to prove than Morley's theorem. That makes it particularly attractive for us; it means that students of classes 11-12 would be able to understand the proof without much difficulty.

You may be puzzled by the name of this theorem: *Napoleon's theorem*. Which Napoleon is this, you may wonder. Well, the reference is indeed to Napoleon Bonaparté, who was known to be a patron of both the sciences and mathematics; he generally moved around with an entourage of scientists and mathematicians, including scholars as established as Fourier,

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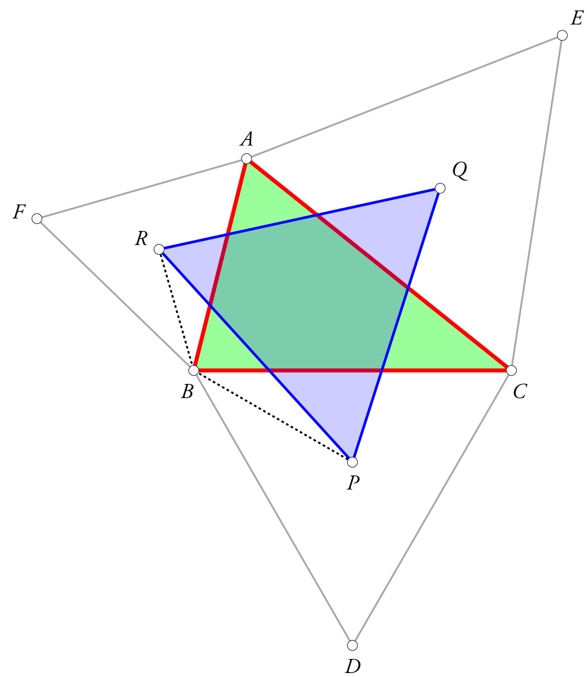


Figure 1

Laplace and Lagrange. The attribution of the theorem to Napoleon may derive from this fact. Other than this, there is no evidence that Napoleon knew of the result that would one day be attributed to him. (It is of course possible that he himself stumbled upon the discovery. Let us not be unfair to him ....)

The statement of the theorem is given below (Box 1).

Napoleon's theorem

Let ABC be an arbitrary triangle. With the three sides of the triangle as bases, construct three equilateral triangles, each one outside $\triangle ABC$. Next, mark the centres P, Q, R of these three equilateral triangles. Napoleon's theorem asserts that $\triangle PQR$ is equilateral, irrespective of the shape of $\triangle ABC$. (See Figure 1.)

Box 1

A trigonometric proof. Perhaps the most straightforward proof of the result is computational, through the use of trigonometry. We consider $\triangle BPR$ and compute the length of side PR, using the cosine rule. In the derivation below, we use the standard short forms for the elements of $\triangle ABC$ (a, b, c for the lengths of the sides, $s = (a + b + c)/2$ for the semi-perimeter, Δ for the area of the triangle, and so on). Here are the steps. Using the cosine rule in $\triangle BPR$ we get:

$$PR^2 = BP^2 + BR^2 - 2BP \cdot BR \cdot \cos \measuredangle RBP.$$

Since $\measuredangle RBP = B + 60°$, we have:

$$\cos \measuredangle RBP = \cos(B + 60°) = \frac{\cos B}{2} - \frac{\sqrt{3}\,\sin B}{2}.$$

We also have:

$$BP = \frac{2}{3} \times \text{altitude of triangle } DCB = \frac{2}{3} \times \frac{\sqrt{3}}{2} a = \frac{a}{\sqrt{3}},$$

$$BR = \frac{2}{3} \times \text{altitude of triangle } FBA = \frac{2}{3} \times \frac{\sqrt{3}}{2} c = \frac{c}{\sqrt{3}}.$$

Hence:

$$\begin{aligned} PR^2 &= \frac{a^2}{3} + \frac{c^2}{3} - \frac{2ac}{3}\left(\frac{\cos B}{2} - \frac{\sqrt{3}\,\sin B}{2}\right) \\ &= \frac{a^2}{3} + \frac{c^2}{3} - \frac{ac\cos B}{3} + \frac{ac\sin B}{\sqrt{3}}. \end{aligned}$$

Next we have, using the cosine rule:

$$2ac\cos B = c^2 + a^2 - b^2.$$

Also, one of the formulas for the area of a triangle ("area of triangle equals half the product of any two sides and the sine of the included angle") yields:

$$\Delta = \frac{1}{2} ac \sin B.$$

Hence:

$$PR^2 = \frac{a^2}{3} + \frac{c^2}{3} - \frac{c^2 + a^2 - b^2}{6} + \frac{2\Delta}{\sqrt{3}},$$

and this simplifies to:

$$PR^2 = \frac{a^2 + b^2 + c^2}{6} + \frac{2\Delta}{\sqrt{3}}.$$

The crucial aspect of the above result is that the expression for PR^2 is ***symmetric*** in a, b, c. This tells us that we will get exactly the same expression for QR^2 as well as PQ^2. It follows that $PQ = QR = RP$, i.e., $\triangle PQR$ is equilateral. □

This proof is purely computational. Such proofs are not to the liking of all readers, but they certainly accomplish whatever is desired; we cannot fault them in any way. So while we are at it, we give another such proof!

A proof using complex numbers. We use the following elegant result which comes from the geometry of complex numbers. Let A, B, C be three distinct points such that in $\triangle ABC$, the direction $[A, B, C, A]$ is counterclockwise (see Figure 2). Let a, b, c denote the complex numbers which represent the points A, B, C respectively. Then if $\triangle ABC$ is equilateral, we have:

$$a + bw + cw^2 = 0, \tag{1}$$

where $w = \cos 120^\circ + i \sin 120^\circ$ is that complex cube root of unity which has argument 120°. Since $w^3 = 1$, relation (1) can be written in the following equivalent forms:

$$c + aw + bw^2 = 0, \quad b + cw + aw^2 = 0.$$

To see why these relations are true, recall the geometrical meaning of multiplication by a complex number. Multiplication by $\cos\theta + i\sin\theta$ accomplishes rotation around the origin through an angle θ, in a counterclockwise direction; so in particular:

- multiplication by w accomplishes rotation by 120° in a counterclockwise direction;

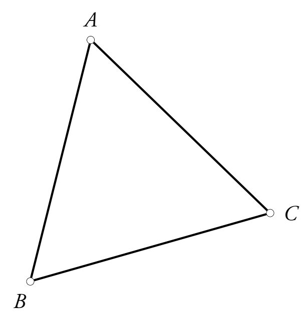


Figure 2

- multiplication by w^2 accomplishes rotation by 240° in a counterclockwise direction, which is the same as rotation by 120° in a clockwise direction;
- multiplication by $-w^2$ accomplishes rotation by 60° in a counterclockwise direction (because $-w^2 = \cos 60^\circ + i\sin 60^\circ$);
- multiplication by $-w$ accomplishes rotation by 60° in a clockwise direction.

In each case, the rotation is about the origin.

Referring to Figure 2, since rotation about B through a 60° angle in the counterclockwise direction takes C to A, it follows that

$$a - b = -w^2(c - b).$$

This relation may be written as:

$$a - \left(1 + w^2\right) b + w^2 c = 0.$$

Since $1 + w^2 = -w$, this yields $a + wb + w^2c = 0$, as claimed. □

The proof as presented can be reversed at every step; please check that this is so. This means that the following converse statement is true as well: if a, b, c are distinct complex numbers such that $a + wb + w^2c = 0$, then $\triangle ABC$ with vertices A, B, C corresponding to a, b, c (respectively) is equilateral. (More can be said: the orientation of the cycle $[A, B, C, A]$ will be *counterclockwise*; but generally we are not concerned by this part of the result.) It is this converse statement which comes of use in proving Napoleon's theorem.

The reader will no doubt notice a lack of symmetry in the relations,

$$a + wb + w^2c = 0, \quad c + aw + bw^2 = 0, \quad b + cw + aw^2 = 0; \tag{2}$$

namely, they do not treat a, b, c 'equally.' But it is easy to see the reason for the lack of symmetry: it stems from the assumption that when we traverse the cycle $[A, B, C, A]$, we travel in a *counterclockwise* direction. Once we realise this, we see immediately that the following result is true as well: If $\triangle ABC$ is such that the direction $[A, B, C, A]$ is clockwise, and $\triangle ABC$ is equilateral, then the following equalities must hold:

$$a + bw^2 + cw = 0, \qquad b + cw^2 + aw = 0, \qquad c + aw^2 + bw = 0. \tag{3}$$

As earlier, the converse proposition is true as well. Note again the lack of symmetry in these relations: they do not treat a, b, c equally.

If we bring together these two non-symmetric results, we obtain a result which is fully symmetric in a, b, c. We have seen that if $\triangle ABC$ is equilateral and the direction $[A, B, C, A]$ is counterclockwise, then $a + bw + cw^2 = 0$; and if the direction $[A, B, C, A]$ is clockwise, then $a + bw^2 + cw = 0$. These two statements taken together imply the following: if $\triangle ABC$ is equilateral, then

$$\left(a + bw + cw^2\right) \cdot \left(a + bw^2 + cw\right) = 0.$$

Moreover, the converse statement must be true as well; if the above product is 0, then one of the two bracketed terms must be 0, hence $\triangle ABC$ must be equilateral. If we multiply out the above product, the result comes as a surprise. We obtain:

$$a^2 + b^2 + c^2 - ab - bc - ca = 0. \tag{4}$$

Note that we have obtained a relation which is fully symmetric in a, b, c! So we obtain the following result as a bonus: if a, b, c are such that $a^2 + b^2 + c^2 = ab + bc + ca$, then $\triangle ABC$ with vertices A, B, C at the points represented by a, b, c is equilateral. Box 2 summarises all these results.

Conditions that make a triangle equilateral

Let A, B, C be three distinct points in the coordinate plane, and let the complex numbers representing these points be a, b, c. The following claims may now be made:

- If the direction $[A, B, C, A]$ is counterclockwise, then $\triangle ABC$ is equilateral if and only if $a + bw + cw^2 = 0$; equivalently, if and only if each of the following equalities holds:

$$a + bw + cw^2 = 0, \quad c + aw + bw^2 = 0, \quad b + cw + aw^2 = 0.$$

- If the direction $[A, B, C, A]$ is clockwise, then $\triangle ABC$ is equilateral if and only if $a + bw^2 + cw = 0$; equivalently, if and only if each of the following equalities holds:

$$a + bw^2 + cw = 0, \quad c + aw^2 + bw = 0, \quad b + cw^2 + aw = 0.$$

- $\triangle ABC$ is equilateral if and only if $\left(a + bw + cw^2\right) \cdot \left(a + bw^2 + cw\right) = 0$, i.e., if and only if

$$a^2 + b^2 + c^2 - ab - bc - ca = 0.$$

- The above may also be written as:

$$\triangle ABC \text{ equilateral} \iff (a-b)^2 + (b-c)^2 + (c-a)^2 = 0.$$

Box 2

Before moving on, we note that equality (4) can be written in the following still more elegant form:

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 0. \tag{5}$$

If a, b, c are ***real*** numbers, then equality (5) holds if and only if $a = b = c$. It is striking that the shift from the real domain to the complex domain can result in so dramatic a change in conclusion.

Let us use these findings to prove Napoleon's theorem. We refer to Figure 1 and use lower case letters to denote the complex numbers representing the respective points (a for A, b for B, ...). Noting carefully the orientations of the various triangles, we obtain the following:

$$d + wc + w^2 b = 0,$$

$$c + we + w^2 a = 0,$$

$$b + wa + w^2 f = 0.$$

We also have (since P, Q, R are the centroids of the respective triangles):

$$3p = b + d + c, \quad 3q = c + e + a, \quad 3r = a + f + b.$$

To prove that $\triangle PQR$ is equilateral, we must prove that $p + wq + w^2 r = 0$. Hence we must prove that

$$(b + d + c) + w(c + e + a) + w^2(a + f + b) = 0.$$

This is equivalent to proving that:

$$\left(d + wc + w^2 b\right) + \left(c + we + w^2 a\right) + \left(b + wa + w^2 f\right) = 0.$$

But this is immediate, since each of the bracketed terms is itself equal to 0. Hence the conclusion follows, that $\triangle PQR$ is equilateral.

Box 3 describes the basic strategy followed in this proof. □

Strategy for proving Napoleon's theorem (outline)

- We use relation (1) which connects the complex numbers representing the vertices of an equilateral triangle and apply it to the three constructed equilateral triangles.
- Then we find expressions for the centroids of the three equilateral triangles in terms of the complex numbers representing the vertices of the triangle.
- Finally, we use the converse of relation (1) to arrive at the desired result.

Box 3

Yet another computational proof. Another approach, involving more manipulations than the one above, is to obtain explicit expressions for p, q, r. We first obtain an expression for d. Since rotation about C through 60° (counterclockwise) takes B to D,

$$d - c = \left(-w^2\right)(b - c),$$

which yields $d = -w^2 b + \left(1 + w^2\right) c$, i.e.,

$$d = -w^2 b - wc,$$

since $w^2 = -1 - w$. We similarly get expressions for e and f. Since P is the centroid of $\triangle BCD$, we have: $3p = b + c + d = \left(1 - w^2\right) b + (1 - w)c$, and similarly for q and r. Thus:

$$3p = \left(1 - w^2\right) b + (1 - w)c,$$

$$3q = \left(1 - w^2\right) c + (1 - w)a,$$

$$3r = \left(1 - w^2\right) a + (1 - w)b.$$

We need to verify that $p + wq + w^2 r = 0$. The coefficient of a in $3\left(p + wq + w^2 r\right)$, obtained by adding suitable multiples of the above three equations, is:

$$\left(w - w^2\right) + \left(w^2 - w^4\right) = w - w^2 + w^2 - w = 0,$$

and similarly for the coefficients of b and c. Hence $p + wq + w^2 r = 0$, and it follows that $\triangle PQR$ is equilateral. □

Closing remark 1. It is worth drawing attention to the use of the word 'symmetry' and words such as 'similarly' which also indicate a kind of symmetry. For most younger students, the word *symmetry* has a strongly geometrical connotation. Here, though we are proving a geometric theorem, our methods have been heavily algebraic; yet we have made use of symmetry at various points: not geometrical symmetry, but algebraic symmetry, the symmetry of symbols, in which we implicitly make use of the fact that nature does not have preferences between the sides of a triangle. This lack of preference naturally carries over to the symbols denoting the sides of the triangle.

Closing remark 2. There are yet other proofs of Napoleon's theorem. In particular, there are proofs that use no computations whatever; rather, they use transformations. There are also striking generalisations of Napoleon's theorem. There is even a result which is a kind of converse to Napoleon's theorem! We shall study all these and more in subsequent parts of this article.

The cube roots of unity: a short tutorial

By the term *cube roots of unity*, we mean the solutions of the cubic equation $x^3 = 1$ over the complex numbers. As the equation is of degree 3, it has 3 roots. We can get them explicitly by noting that: (i) $x - 1$ is a factor of $x^3 - 1$, (ii) the remaining factor is quadratic:

$$x^3 - 1 = (x-1)\left(x^2 + x + 1\right).$$

Hence the solutions are: $x = 1$, and

$$x = \frac{-1 \pm \sqrt{1-4}}{2}, \quad \text{i.e.,} \quad x = \frac{-1 + i\sqrt{3}}{2}, \quad x = \frac{-1 - i\sqrt{3}}{2},$$

where $i = \sqrt{-1}$. The latter two roots are non-real complex numbers. Note that they are conjugates of each other. It is traditional to denote the first one (with positive imaginary part) by w; then the second one is its conjugate $\overline{w}$. We list below a number of properties of these complex numbers. They frequently come of use in geometric applications.

(1) $|w| = 1$; $\left|w^2\right| = 1$; $\arg w = 120^\circ$; $\arg w^2 = -120^\circ$.

(2) $\overline{w} = w^2$ and $w = (\overline{w})^2$; that is, each non-real cube root of unity is the square of the other one. In the same way, each non-real cube root of unity is the reciprocal of the other one. So the three cube roots of unity may be expressed as $1, w, w^2$ or as $1, w, 1/w$.

(3) $1 = w^3 = w^6 = w^9 = w^{12} = w^{15} = \cdots$.

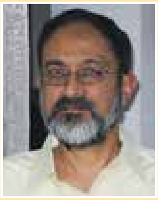
(4) $w = w^4 = w^7 = w^{10} = w^{13} = w^{16} = \cdots$.

(5) $w^2 = w^5 = w^8 = w^{11} = w^{14} = w^{17} = \cdots$.

(6) $1 + w + \overline{w} = 0$; otherwise put, $1 + w + w^2 = 0$.

(7) The three cube roots of -1 are: -1, $-w$, $-w^2$.

(8) The six sixth roots of unity are: 1, w, w^2, -1, $-w$, $-w^2$.



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