DIVISIBILITY BY 7

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This article deals with a simple test for divisibility by 7 for natural numbers having a minimum of four digits. Here, a case of a six-digit number is proved initially and similar proofs follow for other higher-digit numbers.

Six-digit numbers
Let n be a six-digit natural number, $n = \overline{abcdef}$. That is,

$$n = 100000a + 10000b + 1000c + 100d + 10e + f. \qquad (1)$$

Theorem. 7 divides n if and only if 7 divides $|p - q|$, where p is the number formed by the first three digits of n and q is the number formed by the last three digits of n, i.e.,

$$\left.\begin{aligned} p &= \overline{abc} = 100a + 10b + c, \\ q &= \overline{def} = 100d + 10e + f. \end{aligned}\right\} \qquad (2)$$

Proof. Let $n = \overline{abcdef}$. Assume that 7 divides $|p - q|$; we shall prove that 7 divides n.

We are told that 7 divides $|p - q|$; hence $p - q = 7k$ where k is an integer. This yields:

$$p - q = (100a + 10b + c) - (100d + 10e + f) = 7k. \qquad (3)$$

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Next,

$$
\begin{aligned}
n &= 100000a + 10000b + 1000c + (100d + 10e + f) \\
&= 100000a + 10000b + 1000c + (100a + 10b + c) - 7k, \text{ (from equation (3))} \\
&= 100100a + 10010b + 1001c - 7k = 1001(100a + 10b + c) - 7k \\
&= 143 \times 7(100a + 10b + c) - 7k.
\end{aligned}
$$

From this, we see that n is a multiple of 7. That is, 7 divides n. Next, assume that 7 divides n; to prove that 7 divides $|p - q|$, we follow much the same steps as used above (in a slightly different order; please fill in the steps).

Corollary. If 7 divides the number $\overline{abcdef}$, then 7 divides the number $\overline{defabc}$.

Examples

(1) Let $n = 976213$; then $p = 976$ and $q = 213$. Hence $p - q = 976 - 213 = 763$.

Since 7 divides 763, it follows that 7 divides 976213. (The quotient in the division is 139459.)

(2) Let $n = 123782$. Here $p = 123$ and $q = 782$. Hence $|p - q| = |782 - 123| = 659$.

Now, 7 does not divide 659. Hence, 7 does not divide 123782.

Remarks

- Similar statements are true for four-digit numbers, five-digit numbers and higher-digit numbers. The only condition to be kept in mind, in all cases, is that the q-block should be made up of the last three digits from the right side. Then p will be formed by all the digits from the left except the last three, as mentioned. That is, q is the number made up of the last three digits of the number, and p is the number made up of the remaining digits. The statement made in the theorem now holds, regardless of how many digits p has: *The original number is divisible by 7 if and only if $p - q$ is divisible by 7. For a short proof of this claim, please see Box 1.*
- Interestingly, all the cyclic numbers formed from the number 976213 obey the statement given above; hence, all these numbers are divisible by 7. For example, 762139 is divisible by 7.

Some more examples

Shown below are some more examples to illustrate this method:

Four-digit numbers:

- Let $n = 1239$; here, $p = 1$, $q = 239$, $|p - q| = 239 - 1 = 238$. Now, 7 divides 238. Hence 7 divides 1239.
- Let $n = 9321$; here, $p = 9$, $q = 321$, $|p - q| = 321 - 9 = 312$. Now, 7 does not divide 312. Hence 7 does not divide 9321.

Five-digit numbers:

- Let n = 34426; here, p = 34, q = 426, $|p - q|$ = 426 – 34 = 392. Now, 7 divides 392. Hence 7 divides 34426.
- Let n = 12345; here, p = 12, q = 345, $|p - q|$ = 345 – 12 = 333. Now, 7 does not divide 333. Hence 7 does not divide 12345.

Nine-digit numbers:

- Let n = 258469232; here, p = 258469, q = 232, $|p - q|$ = 258237. If we repeat the same step for the new number, we get 258–237 = 21. Now, 7 divides 21. Hence 7 divides 258237. Hence 7 divides 258469232.

Proof of the General Claim - Box 1

Given a positive integer n, let q be the number made up of the last three digits of n, and let p be the number made up of the remaining digits. Then the following is true: n is divisible by 7 if and only if $p - q$ is divisible by 7. With p and q as defined above, we have the relation $n = 1000p + q$. For example,

$$2345 = (2 \times 1000) + 345,$$

$$12345 = (12 \times 1000) + 345.$$

Hence the following is equivalent to the claim made above:

$$7 \text{ divides } 1000p + q \iff 7 \text{ divides } p - q. \tag{5}$$

Statement (5) may be proved as follows.

Proof. Let $a = 1000p + q$ and $b = p - q$. Then $a + b = 1001p$, which is a multiple of 7 (since $1001 = 7 \times 11 \times 13$). Hence if a is a multiple of 7, so must be b; and if b is a multiple of 7, so must be a. This is exactly the claim made above.

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