Problem Corner

Problems for the SENIOR SCHOOL

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Problem VI-3-S.1

The numbers

$$\frac{1}{1}, \frac{1}{2}, \dots, \frac{1}{2017}$$

are written on a blackboard. A student chooses any two numbers from the blackboard, say x and y, erases them and instead writes the number $x + y + xy$. She continues to do so until there is just one number left on the board. What are the possible values of the final number?

Problem VI-3-S.2

The numbers $1, 2, 3, \dots, n$ are arranged in a certain order. One can swap any two adjacent numbers. Prove that after performing an *odd* number of such operations, the arrangement of the numbers thus obtained will differ from the original one.

Problem VI-3-S.3

At each of the eight corners of a cube, write $+1$ or -1. Then, on each of the six faces of the cube, write the product of the numbers written at the four corners of that face. Add all the fourteen numbers so written down. Is it possible to arrange the numbers $+1$ and -1 at the corners initially in such a way that this final sum is zero?

Keywords: parity, integers, combinatorics, polynomials, identities

Problem VI-3-S.4

At a party, it was observed that each person knew exactly twenty others. Also, for each pair of persons who knew one another, there was exactly one other person whom they both knew. Further, for each pair of persons who did not know one another, there were exactly six other persons whom they both knew. Also, if A and B were present in the party and A knew B, then B knew A. Determine, with proof, the number of people at the party.

Problem VI-3-S.5

Suppose there are k teams playing a round-robin tournament; that is, each team plays against every other team. Assume that no game ends in a draw. Suppose that the i-th team loses l_i games and wins w_i games. Show that

$$\sum_{i=1}^{k} l_i^2 = \sum_{i=1}^{k} w_i^2.$$

Solutions to Problems in Issue-VI-2 (July 2017)

Solution to problem VI-2-S.1

A teacher wrote the quadratic $x^2 + 10x + 20$ on the board. Then each student either increased by 1 or decreased by 1 either the constant or the linear coefficient. Finally $x^2 + 20x + 10$ appeared. Did a quadratic expression with integer zeros necessarily appear on the board in the process? [From *Polynomials* by Ed Barbeau]

Solution. The key observation to make, which solves the problem, is that the quadratic expression $x^2 + (k+1)x + k$ has integer zeros for any integer value of k. (Namely: its zeros are $-k$ and -1.)

Suppose that at some stage of the process, the quadratic expression on the board is $x^2 + ax + b$. What happens at the next stage? The ordered pair (a, b) is replaced by the ordered pair (a', b') which is one of $(a+1, b)$, $(a, b+1)$, $(a-1, b)$, $(a, b-1)$. Now observe that $a' - b' = a - b \pm 1$. That is, the value of $a - b$ changes by 1 at each stage (it either decreases or increases) . Initially $a - b = -10$; at the end $a - b = 10$. So the difference between a and b climbs from -10 to 10, changing by ± 1 at each step. If we can guarantee that $a - b = 1$ at some stage, then we will be done. But that will certainly happen at some stage, as each change is ± 1. So the answer is: a quadratic expression with integer zeros does necessarily appear on the board, at some stage.

Solution to problem VI-2-S.2

Let $p(t)$ be a monic quadratic polynomial. Show that, for any integer n, there exists an integer k such that $p(n)p(n+1) = p(k)$. [From *Polynomials* by Ed Barbeau]

Solution. Let $p(t) = t^2 + bt + c$. Then $p(n) = n^2 + bn + c$ and $p(n+1) = (n+1)^2 + b(n+1) + c$. Then

$$\begin{aligned}(p(n) - c)(p(n+1) - c) &= n(n+b)(n+1)(n+1+b)\\ &= (n(n+1) + bn)(n(n+1) + bn + b)\\ &= m(m+b),\end{aligned}$$

where $m = n(n+1) + bn$. But

$$p(n) + p(n+1) = 2(n^2 + n + bn) + 1 + b + 2c = 2m + 1 + b + 2c.$$

Therefore

$$\begin{aligned} p(n)p(n+1) &= m^2 + (b+2c)m + c(1+b+c) \\ &= (m+c)^2 + b(m+c) + c \\ &= p(m+c) = p(n(n+1) + bn + c). \end{aligned}$$

Thus $k = n(n+1) + bn + c$. This value of k "works."

Solution to problem VI-2-S.3
Prove that the product of four consecutive positive integers cannot be equal to the product of two consecutive positive integers. [From Round 1, British Mathematical Olympiad, 2011]

Solution. Suppose there exist positive integers m and n such that

$$m(m+1) = n(n+1)(n+2)(n+3).$$

Then

$$\begin{aligned} m(m+1) + 1 &= (n^2 + 3n)(n^2 + 3n + 2) + 1 \\ &= u(u+2) + 1 = (u+1)^2, \end{aligned}$$

where $u = n^2 + 3n$. However, $m^2 < m^2 + m + 1 < (m+1)^2$. That is, the number $m^2 + m + 1$ lies between squares of two consecutive positive integers and hence cannot be a square. This means that the stated equality, that $m(m+1) = n(n+1)(n+2)(n+3)$ for some two positive integers m, n, cannot take place.

Solution to problem VI-2-S.4
Find all integers n for which $n^2 + 20n + 11$ is a perfect square. [From Round 1, British Mathematical Olympiad, 2011]

Solution. Let $n^2 + 20n + 11 = y^2$. Then $(n+10)^2 - y^2 = 89$, hence

$$(n + 10 + y)(n + 10 - y) = 89.$$

Since 89 is a prime number, the only ways that it can be written as a product of two integers are $\pm 89 \times \pm 1$ and $\pm 1 \times \pm 89$. So the following cases arise.

(i) Case 1: $n + 10 + y = 89$, $n + 10 - y = 1$, giving $n = 35$.

(ii) Case 2: $n + 10 + y = 1$, $n + 10 - y = 89$, giving $n = 35$.

(iii) Case 3: $n + 10 + y = -1$, $n + 10 - y = -89$, giving $n = -55$.

(iv) Case 4: $n + 10 + y = -89$, $n + 10 - y = -1$, giving $n = -55$.

So there are just two possible values of n, namely -55 and 35.

Solution to problem VI-2-S.5

Find all integers x, y, z such that $x^2 + y^2 + z^2 = 2(yz + 1)$ and $x + y + z = 4018$. [From Round 1, British Mathematical Olympiad, 2009]

Solution. From the first equation we have $x^2 + (y - z)^2 = 2$. Therefore $x = \pm 1$ and $y - z = \pm 1$.

(i) Case 1: $x = 1$. This leads to $y + z = 4017$ and $y - z = \pm 1$. Thus $(x, y, z) = (1, 2009, 2008), (1, 2008, 2009)$.

(ii) Case 2: $x = -1$. This leads to $y + z = 4019$ and $y - z = \pm 1$. Thus $(x, y, z) = (-1, 2010, 2009), (-1, 2009, 2010)$.

These are the only solutions in integers of the given pair of equations.