The Rascal Triangle

A Rascal full of Surprises!

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As a teacher, one is always looking for so-called 'non-standard' problems. These should be based on material that has been taught, and yet be neither trivial nor too hard. This article illustrates an example of such a non-standard problem. Reading the article on the Rascal Triangle [1], I felt it would fit the bill, given that it was discovered by students in the first place. As an introduction to the problem, I put down the following six rows (Figure 1), and titled it

'The Rascal Triangle'

1

1 1

1 2 1

1 3 3 1

1 4 5 4 1

1 5 7 7 5 1

Figure 1

Keywords: *Pascal triangle, investigation, pattern, prediction, successive differences*

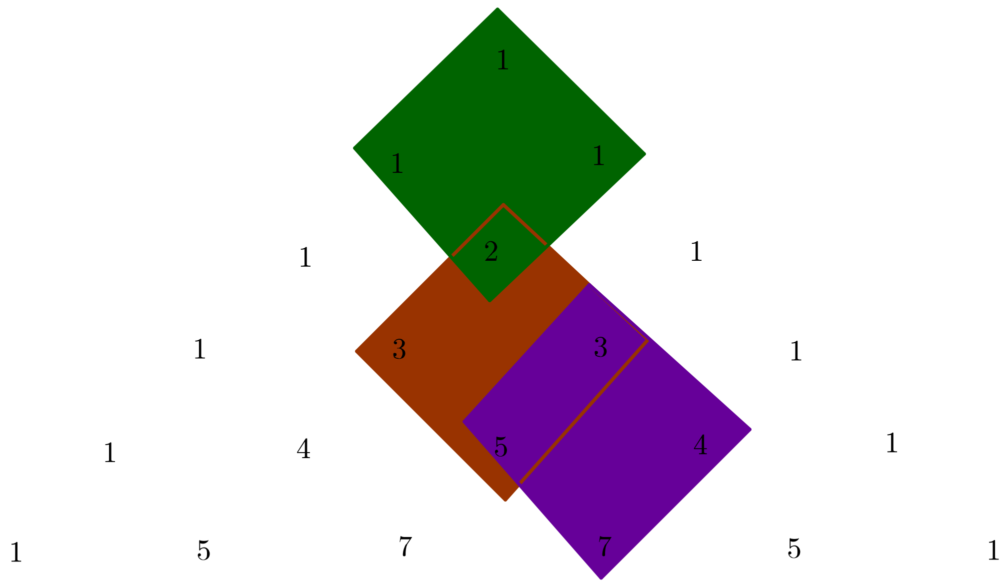

Figure 2

Since my students were already familiar with the Pascal triangle, I told them the story of the Rascal triangle and how it was discovered, and asked my students to find the pattern for the next few rows and come up with a rule to generate rows, somewhat like the rule for Pascal's triangle.

As a hint, I drew Figure 2 and explained (following the lead from [1]) that just as we can think of the rule generating elements in the Pascal triangle to be based on a triangle, we can think of the rule generating the elements of the Rascal triangle to be based on a diamond.

My hope was that my students would rediscover the formula that the American students had found. Knowing their familiarity with finding formulae for sequences with constant differences, I also asked them to find the formula for the r^{th} entry in the m^{th} row. I was not sure how much progress they would make in this regard, and thought I would have to explain the method the article uses to derive the formula.

I did not realize that there would be many surprises awaiting me! The first was that three of my students discovered an entirely new 'diamond rule'. Here is a description of their discovery in Ishaan and Maya's own words:

"Given some time to stare at it, we suggested that the number x in the diamond (Figure 3) could be obtained by the formula $x = b + c + 1 - a$.

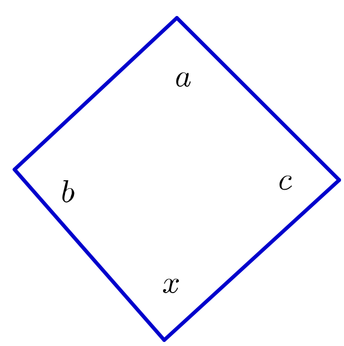

Figure 3

While arriving at the answer seemed to be an almost instantaneous process, we had very different approaches to finding the formula. One approach was to try various arithmetic processes (addition, subtraction, multiplication and division) indiscriminately! The other approach was to try addition first driven by the belief that it is the only guaranteed method by which one would always get an integer value for x using integers a,b,c.

Our formula appeared to work for all the values in the entries of Rascal triangle given to us and it surprised our math teacher, because it did not resemble the one that the students who originally created the Rascal triangle had come up with."

The second surprise was that Rishabh came up with a different approach to derive the formula for the r^{th} entry in the m^{th} row. To begin with almost all students recognized the pattern among the diagonal rows [1], so they were able to generate rows of the Rascal triangle at will.

Here is Rishabh in his own words.

"I started by looking (Figure 4) at the first difference of all the rows:

I saw that for any given row of first differences, successive elements differ by –2. Also the first element in each of the blue rows can be found by subtracting 2 from the number of the row, that is, for the m^{th} row, the first element of the corresponding blue row of successive differences will be m–2. Using the same notation as [1] to denote the r^{th} element in the m^{th} row to be Entry (m, r). Writing $1 = m - (m - 1)$, I got the following list:

Entry $(m, 1) = m - (m - 1)$

Entry $(m, 2) = m - (m - 1) + (m - 2)$

Entry $(m, 3) = m - (m - 1) + (m - 2) + (m - 4)$

Entry $(m, 4) = m - (m - 1) + (m - 2) + (m - 4) + (m - 6)$

Entry $(m, 5) = m - (m - 1) + (m - 2) + (m - 4) + (m - 6) + (m - 8)$

Simplifying:

Entry $(m, 1) = m - (m - 1) = 0m + 1$

Entry $(m, 2) = m - (m - 1) + (m - 2) = 1m - 1$

Entry $(m, 3) = m - (m - 1) + (m - 2) + (m - 4) = 2m - 5$

Entry $(m, 4) = m - (m - 1) + (m - 2) + (m - 4) + (m - 6) = 3m - 11$

Entry $(m, 5) = m - (m - 1) + (m - 2) + (m - 4) + (m - 6) + (m - 8) = 4m - 19$

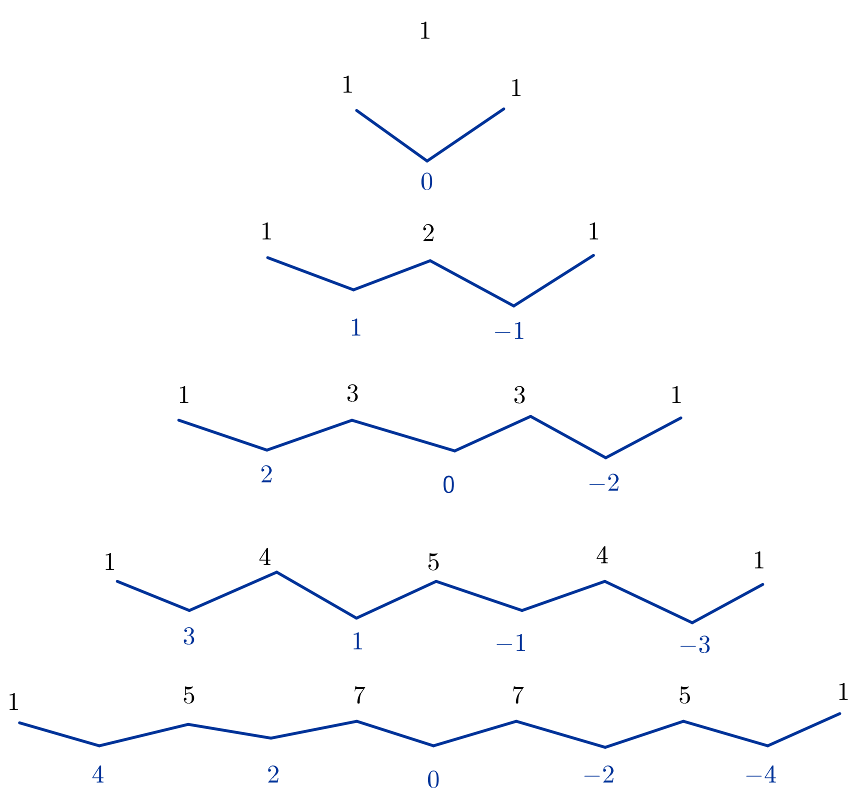

Figure 4

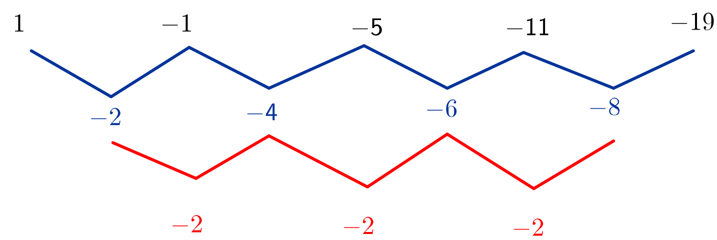

Figure 5

It is easy to see that the coefficient of m in each entry is simply $r - 1$. I tried to find a formula for the terms $1, -1, -5, -11, -19, \ldots$.

When I examined the differences amongst the terms (Figure 5), I found the second difference to be a constant. Using the standard technique for finding the formula for sequences whose second difference is a constant, the formula for this sequence is given by $-r^2 + r + 1$. Therefore:

$$\text{Entry } (m, r) = m(r-1) - r^2 + r + 1 = (r-1)(m-r) + 1.$$

Here $m = 1, 2, 3 \ldots$ and $r = 1, 2, \ldots m$."

Finishing up. To finish up, we need to show two things:

1. The formula we have found for the r^{th} entry of the m^{th} row is the same as that derived in the original article
2. Our new diamond rule is equivalent to the diamond rule discovered by the students from America.

Recall that the formula for the k^{th} element of the n^{th} row given in the original is Entry $(n, k) = k(n - k) + 1$, where $n = 0, 1, 2, \ldots$ and $k = 0, 1, 2 \ldots n - 1$. It is obvious that all we need to do is set $m = n + 1$ and $r = k + 1$ to show the two are equivalent.

To establish the second we prove the following theorem (replace with similar to [1]). Here we would like to thank Dr Shirali for pointing us in this direction. We will use Rishabh's formula for the r^{th} element of the m^{th} row to do so!

Theorem. *For the sub-array:*

	a	
b		c
	x	

the new entry x is given by:

$$x = b + c + 1 - a.$$

Proof. Examining the framed box above, all we need to verify is the following:

Entry $(m + 1, r + 1)$ = Entry (m, r) + Entry $(m, r + 1) + 1$ − Entry $(m - 1, r)$,

i.e.,

$$r(m-r) + 1 = \big((r-1)(m-r) + 1 + \big(r(m-r-1) + 1 + 1 - \big((r-1)(m-r-1) + 1\big).$$

The verification takes but a moment!

References

1. $\mathscr{C}\otimes\mathscr{M}\alpha\mathscr{C}$, "The Rascal Triangle". *At Right Angles*, Volume 5, No.1 , March 2016

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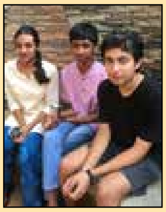

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SUMS OF **CONSECUTIVE** NUMBERS THAT YIELD POWERS OF 9

$$1 = 9^0$$

$$2+3+4 = 9^1$$

$$5+6+7+8+9+10+11+12+13 = 9^2$$

$$14+15+16+17+\cdots+37+38+39+40 = 9^3$$

$$41+42+43+44+\cdots+118+119+120+121 = 9^4$$

And so on.

REFERENCES

[1] www.webalice.it/francesco.daddi [2] www.facebook.com/matematicadaddi