Through the Symmetry Lens

Part II - Wallpaper Patterns and Symmetry Around Us

Features

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This is the second part of a two-part article whose aim is to familiarise the reader with both the mathematical concept and an intrinsic idea of symmetry. The first part of the article concentrated on a 'working definition' of symmetry and also laid the mathematical base to understand symmetry. It discussed symmetries of figures that can be drawn on a sheet of paper and of a particular type of infinite pattern called a *strip pattern* or *a frieze pattern.*

In this part we will concentrate on another infinite two-dimensional pattern called the *wallpaper pattern* and also explore aspects of symmetry in the everyday objects around us. For ease, we reiterate the 'working definition' of symmetry here.

Intuitively, symmetry can be thought of as an action performed on an object, which leaves the object looking exactly the same and occupying the exact same space as before. If a person closes her eyes while the action is being performed, she will not know that any action has been performed.

Objects that can be drawn on a sheet of paper (finite planar objects) can have only two kinds of symmetries: rotations and possibly

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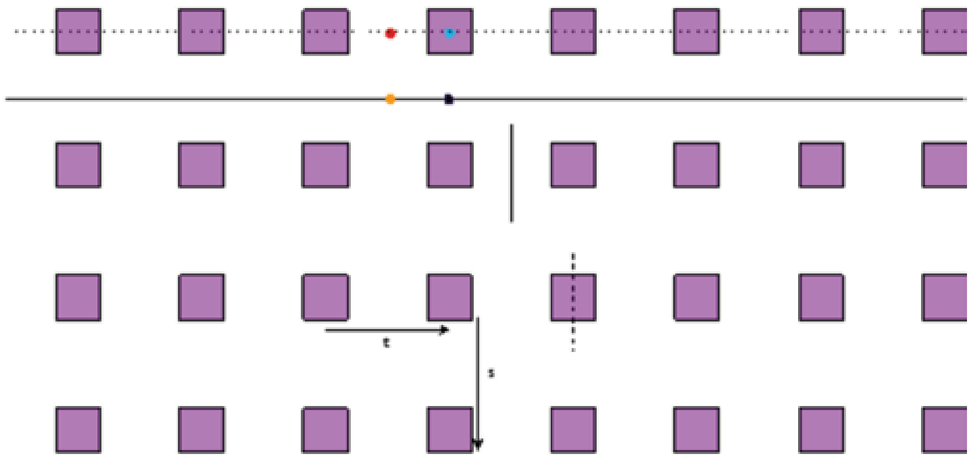

Figure 1

reflections. A strip pattern has a new kind of symmetry, called *translation symmetry*. It may also have symmetries of the following type: 180-degree rotations, reflections and glide-reflections. Part I of the article discusses all this in some detail and the reader is advised to consult the same. We now explore symmetries of wallpaper patterns or tessellations.

Wallpaper Patterns or Tessellations

As the name suggests this will be an infinite pattern that will spread both in the left-right and top-down directions. One way of creating a wallpaper pattern is by choosing a strip pattern and stacking it at equal intervals on top as well as below. An example of a wallpaper pattern is Figure 1.

The wallpaper pattern in Figure 1 has translations in two different directions indicated by **s** and **t**. It has two types of vertical lines of reflection, namely the dotted type and the solid line, and similarly two different types of horizontal lines of reflection. There are four different types of rotocentres indicated about each of which a 180-degree rotation is possible. (Note that since the length of **t** is different from **s**, the grid created is basically rectangular and not square, so reflections about diagonal lines, or 90-degree rotations about the rotocentres will not give us a symmetry of the wallpaper pattern.) Since we have symmetries that are translations and reflections we will also have glide reflections. The reader may try to mark glide reflections on the wallpaper pattern above.

Classifying wallpaper patterns is much more complicated. There are 17 wallpaper patterns[1] in all. It is believed that the 14th-century Alhambra Palace in Granada, Spain, has all 17 patterns exhibited. The classification in a sense is based on the underlying grid (or lattice) in a wallpaper pattern.

For example, consider the example of the wallpaper pattern illustrated and discussed above. We can think of it being created not only as a strip being repeated at equal intervals in the top-down direction but also in the following way. Think of the plane being covered by a rectangular grid or lattice as shown in Figure 2. (Black lines with circles marking the vertices.) Then replace the

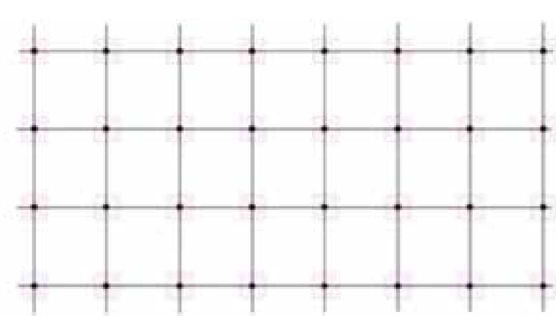

Figure 2

[1] *For more on classification of the wallpaper patterns please see Contemporary Abstract Algebra by Joseph Gallian. The chapter on Frieze Groups and Crystallographic Groups provides algorithms for classifying strip or frieze patterns and also for classifying wallpaper patterns or tessellations.*

vertices of the lattice with squares as shown. So the wallpaper is created by the light pink squares, replacing the black dots and then removing all the grid lines.

It turns out that in any wallpaper pattern, there are only five possible underlying grids or lattices, namely, square, rectangle (non-square), parallelogram (non-square, non-rectangular), equilateral triangle and regular hexagon. Using these and the different positioning of certain basic motifs, 17 wallpaper patterns are created.

Some more examples of wallpaper patterns using the different underlying lattices are given below. The reader is invited to discover which of the following symmetries exist in the examples of wallpaper patterns given below: translations, rotations, reflections and glide-reflections. Note that in the case of translations, the basic distance with directions should be marked. For rotations, rotocentres as well as the angle of rotation should be specified. For reflections, the lines of reflection and for a glide-reflection, the glide line and reflection line ought to be marked.

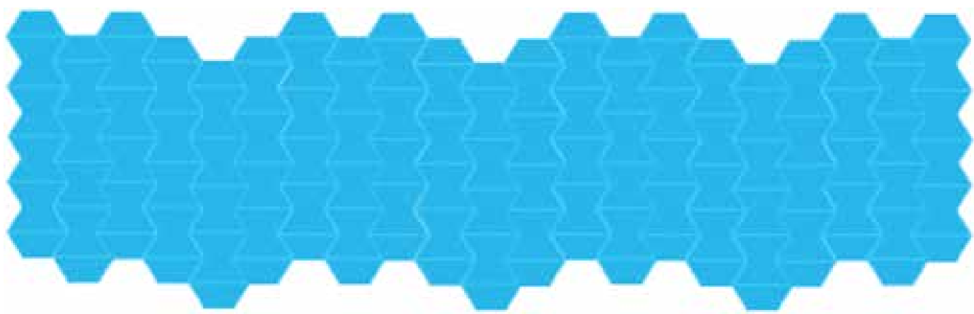

Figure 3 - Wallpaper Pattern based on a Regular Hexagon Lattice

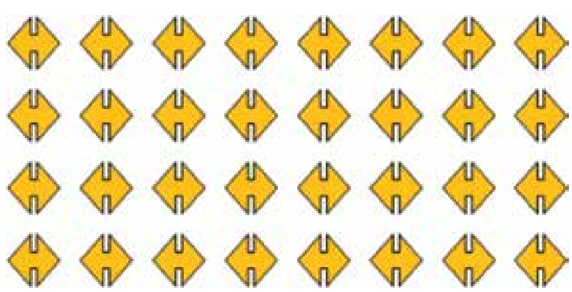

Figure 4 - Wallpaper Pattern based on a Square Lattice

Figure 5 - Wallpaper Pattern based on a Non-rectangular Parallelogram Lattice

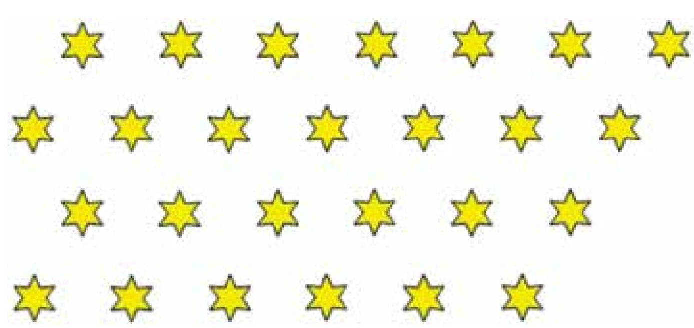

Figure 6 - Wallpaper Pattern based on an Equilateral Triangle Lattice

Exploring Symmetry Around Us

Now that we have an idea of the different types of symmetries that finite planar objects, strip patterns and wallpaper patterns have, we can use our knowledge to view the world around us through the symmetry lens. Of course, we have the ability so far only to make sense of the symmetries of planar objects. Symmetry of three-dimensional objects can also be studied in a similar manner but that is material for a different article.

We provide below examples of objects that we encounter around us often in our everyday lives and explore the underlying symmetry in some of those. Other examples are provided for the reader to peruse and analyse at leisure. One caveat, when exploring symmetry in real life, is that we may need to ignore imperfections or some parts of the scenery in order to appreciate the beauty of symmetry.

Nature abounds in plentiful examples of symmetry. Below are two flowers (Figure 7). The flower on the left has only rotational symmetries and no reflection symmetry. Consider the purple flower. We have marked the flower with *5* brown lines that have an angle of 72-degrees each between successive lines. As can be seen from the figure, the respective lines do not bisect the petals exactly. Indeed if we join the mid-point of the upper tips of the petals, we find that the figure formed is *5*-sided but not a regular pentagon. Using the brown lines we can reason that there are 'approximate' rotations of 0, 72, 144, 216 and 288-degrees. Thus we could classify this flower as of type C_5. (Please refer to Part I of the article in the March 2016 issue of *At Right Angles* for the meanings of these symbols.)

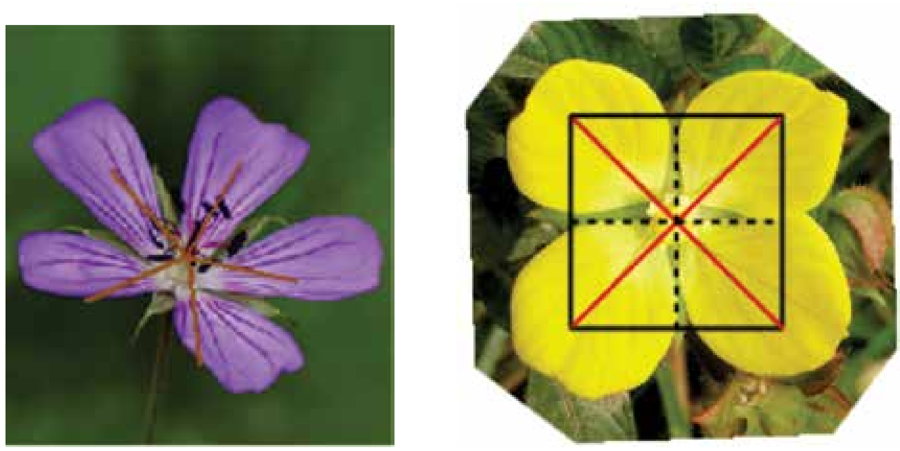

Figure 7

Figure 8

Figure 9

Figure 11

Figure 10

In the case of the yellow flower, the square grid indicates the possibility of 4 rotations of 0, 90, 180 and 270-degrees and 4 reflections; it is therefore of D_4 type. The butterfly in Figure 8 shows reflection symmetry and the 0-degree do-nothing rotational symmetry; it is thus of D_2 type.

We can find examples of strip patterns and wallpaper patterns in the designs on clothes that we wear. Sari borders are often fine examples of strip patterns while the interior provides examples of wallpaper patterns. Some examples are given below.

The pattern shown in Figure 9, which is part of a sari, can be regarded as a wallpaper pattern based on a rectangular grid having translations in two different directions, two different types of vertical lines of reflection and glide reflections. The only rotation possible is the 0-degree or the do-nothing rotation.

The strip in Figure 10 is part of a sari border. On analyzing it we note that there are translations, reflection symmetries in vertical and horizontal direction, 180-degree rotations and glide reflection. It is therefore a Type VII strip pattern.

The four leaves taken together as a motif are part of a cushion cover. It can be seen that this motif will have 4 rotations and 4 reflections and so it has classification type D_4.

The collage in Figure 11 has been made with four images of artwork from the Crafts Museum in Delhi. The top and bottom images should be analysed as strip patterns, the image in the centre right as a wallpaper pattern and the centre left image as a finite planar motif.

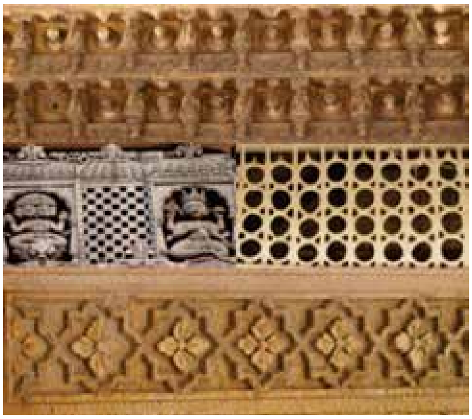

Figure 12

Figure 13

Old monuments, newer buildings, grills on windows, balcony railings, fences, and so on too provide for interesting examples to view via symmetry. The collage in Figure 12 provides examples of this and can be analysed in a manner similar to the earlier collage.

It would be a grave lacuna not to mention the artwork of M C Escher in an article on symmetry. However due to copyright restrictions we cannot reproduce photographs of Escher's work here. It is recommended that the reader peruse his art on the official website www.mcescher.com. We however leave the reader with an image of a floor puzzle inspired by Escher to enjoy at leisure and to discover its secrets via symmetry; see Figure 13.

Bibliography

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