The Magical World of Infinities

Part II

Features

SHASHIDHAR JAGADEESHAN

Introduction

In the previous article we encountered the strange world of infinities, where a lot of our intuitive sense of how infinite sets should behave started breaking down. We saw for example that infinite sets can have subsets which have as many elements as the original set; we also saw our intuition about length breaking down. No matter what the lengths of the two lines are, they ended up having the same number of points. Moreover, all the examples of infinite sets we encountered ended up having the same number of elements. You might naturally assume that there is only one kind of infinity – which is what perhaps you had assumed right from the beginning?

In the last section of Part I of this article (*AtRiA*, March 2016), we had hinted at the possibility of there being different kinds of infinities. If there are, can we mathematically prove they exist? How many different infinities are there really? In this article we will answer these questions.

Recall that the cardinality of a set counts the number of elements it contains. We denote the cardinality of a set X by $|X|$. In some cases we have special symbols denoting the cardinality of sets.

Keywords: *Infinity, set, cardinality, countable, natural number, real number, unit interval, continuum, one-one correspondence, injective function*

For example, $\aleph_0$ represents the cardinality of the set $\mathbb{N}$ of natural numbers, and **c** (called the **continuum**) represents the cardinality of the set $\mathbb{R}$ of real numbers. If an infinite set has cardinality $\aleph_0$, then we say that this set is *countable*. $\aleph_0$ and **c** are examples of cardinal numbers.

At this stage it would be good to introduce some ideas and techniques that we use to compare two infinite sets, since we will be using them quite often. In the previous article we introduced the idea of 1-1 correspondence between two sets X and Y, and said that $|X| = |Y|$ if and only if we can find a 1-1 correspondence between X and Y. In a 1-1 correspondence we have a function which associates every element of X with a unique element of Y, and by inverting this association, every element of Y is associated with a unique element of X. A slightly weaker notion than 1-1 correspondence is the idea of an *injective function* (often referred to as a '1-1 function' as opposed to '1-1 correspondence', but we will use the term injective function to avoid confusion). An injective function $f: X \to Y$ is a function that satisfies the property that if $f(a) = f(b)$, then $a = b$. Notice that in an injective function we cannot be sure that every element in the set Y has a partner in X; however, if an element in the set Y does have a partner in X, then that partner is unique. Figure 1 illustrates an injective function.

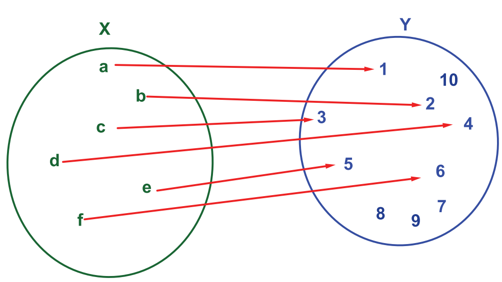

Figure 1

It is clear from Figure 1 that X and Y do not have the same number of elements and $|X| \leq |Y|$, and moreover there is a 1-1 correspondence between X and some subset of Y. Whenever we have $A \subseteq B$, we can construct the obvious injective function $i : A \to B$, which takes each element of A to itself, that is, $i(a) = a$, $\forall a \in A$.

If there is an injective function from a set X to a set Y, we can see that $|X| \leq |Y|$. The **Schroeder-Bernstein Theorem** (sometimes Cantor's name is also added) states that if we have an injective function $f: X \to Y$ and another injective function $g : Y \to X$ (note that f and g need not be inverses of each other), then $|X| = |Y|$.

We are now ready to compare the set $\mathbb{R}$ of real numbers with the set $\mathbb{N}$ of natural numbers and to show that $\aleph_0 < \mathbf{c}$.

Are real numbers countable?

Let us start by comparing the set of all real numbers between 0 and 1 (we denote this set by $(0, 1)$) and the set $\mathbb{N}$ of natural numbers.

We remind readers of the fact that the real numbers have decimal representations. Furthermore, by inserting a string of zeros, we can make it an infinite decimal representation. For example, $\frac{1}{4} = 0.25000\ldots$ (with infinitely many trailing zeros) and $\frac{1}{7} = 0.142857142857142857142857142857142857\ldots.$ The question is, are these representations unique? You might have come across the curious fact that $0.99999\cdots = 1$ (this is really fascinating, if you have not already done so, see if you can prove it for yourself). So it appears that we have two possible decimal representations for some real numbers. It turns out that if we can take care of the case of repeating nines, we can then have unique decimal representations for all real numbers. So, if we decide that we will choose to represent numbers like $0.2999\ldots$ by $0.3000\ldots$, then every member of our set has a unique decimal representation.

Here is Cantor's proof that there are more real numbers between 0 and 1 than there are natural numbers.

It is relatively easy to see that $\aleph_0 \leq |(0, 1)|$. For, consider the following injective function:

$$g : \mathbb{N} \to (0, 1), \quad g(n) = \frac{1}{n+1};$$

then g is clearly an injective function, and from our discussion above we get $\aleph_0 \leq |(0, 1)|$. What we want to show is that the equality is not possible; that is, $\aleph_0 < |(0, 1)|$.

Remember, in order to do this we need to establish that it is *impossible* to have a 1-1 correspondence between the set $(0, 1)$ and $\mathbb{N}$. We do so by assuming the contrary; that is, we assume that there does exist a 1-1 correspondence between these two sets and keep arguing logically, step by step, until something goes wrong! The only reason for something to go wrong could then be that we made an erroneous assumption in the beginning.

If there is a 1-1 correspondence between the set $\mathbb{N}$ of natural numbers and the set $(0, 1)$, we can assign a natural number to each element in $(0, 1)$. Let us denote the number in $(0, 1)$ associated with 1 as a_1, the number associated with 2 as a_2 and so on, allowing us to enumerate the elements of the set $(0, 1)$, using natural numbers thus:

$$(0, 1) = \{a_1,\ a_2,\ a_3,\ \ldots,\ a_n,\ \ldots\}.$$

Let us further denote each element in the above list as a decimal expansion, and let us do it in a manner in which a clear pattern emerges.

$$\begin{aligned}
a_1 &= 0.a_{1,1}\, a_{1,2}\, a_{1,3}\ \ldots\ a_{1,n}\ \ldots\\
a_2 &= 0.a_{2,1}\, a_{2,2}\, a_{2,3}\ \ldots\ a_{2,n}\ \ldots\\
a_3 &= 0.a_{3,1}\, a_{3,2}\, a_{3,3}\ \ldots\ a_{3,n}\ \ldots\\
&\vdots\\
a_n &= 0.a_{n,1}\, a_{n,2}\, a_{n,3}\ \ldots\ a_{n,n}\ \ldots\\
&\vdots
\end{aligned}$$

Now here is where Cantor's brilliance can be seen again. He defines a new element

$$b = 0.b_1\, b_2\, b_3\ \ldots\ b_n\ \ldots$$

in the following manner. Let b_1 be any integer (between 1 and 8) other than $a_{1,1}$; let b_2 be any integer (between 1 and 8) other than $a_{2,2}$; let b_3 be any integer (between 1 and 8) other than $a_{3,3}$; and so on. So b_n is any integer between 1 and 8 other than $a_{n,n}$. Notice that the decimal expansion of b differs from the decimal expansion of a_1 in at least one place (namely, $a_{1,1}$) and similarly from the decimal expansion of a_2 in at least one place and in this way from every element in our list above. (The reason why we do not allow the integers 0 or 9 is to make sure that b does not have all zeros or nines in its decimal expansion.) By this clever construction (which is called Cantor's *diagonalization* technique) we have found an element b such that

$$b \in (0, 1) \text{ and } b \notin \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$$

But this is a contradiction—because we had assumed that every element in the set $(0, 1)$ is accounted for the list $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$! Where did we go wrong? If you go back and check all the steps in our argument, you will find that the mistake was in assuming that there is a one-to-one correspondence between $(0, 1)$ and $\mathbb{N}$. In fact, what Cantor managed to show was that no matter how clever you are, you cannot come up with a 1-1 correspondence between the above two sets, because the moment you do, and you enumerate the elements of $(0, 1)$ using the natural numbers, the diagonalization process guarantees that you will always come up with an element in $(0, 1)$ which is not in the list that you had made! This establishes the fact that the set of real numbers contains more elements than the set of natural numbers, and therefore that $\aleph_0 < |(0, 1)|$ or in other words $(0, 1)$ is not countable.

What about the real numbers, are they countable? In Part I of this article we showed that there is a 1-1 correspondence between the set $(-1, 1)$ and $\mathbb{R}$. We can use a similar argument to show that there is a 1-1 correspondence between $(0, 1)$ and $\mathbb{R}$ and, in fact, between any open interval in the set of real numbers and $\mathbb{R}$. I hope this amazing fact has not slipped by the reader, that the set of real numbers $\mathbb{R}$ and any open interval contained in $\mathbb{R}$ have the same cardinality; namely, the continuum $\mathbf{c}$. Clearly, since $(0, 1)$ is not countable, and $\mathbb{R}$ has the same cardinality as $(0, 1)$, $\mathbb{R}$ is not countable and $\aleph_0 < \mathbf{c}$.

Cantor thus managed to introduce a new infinity! He showed that infinite sets are not all of the same size, that there are different types of infinite sets, which differ because of their sizes. This unleashes a whole set of questions about how many different kinds of infinities there are. It turns out that it is not so straightforward to generate new infinities. In order to illustrate this, we now compare the number of points in a square with the number of points on one of its edges and compare the number of points in a cube with the number of points on one of its edges. Be prepared to be surprised!

Edges, Squares and Cubes

We now compare the number of points in a square with the number of points on one of its edges. Our intuition tells us that since we are comparing objects in different dimensions, clearly the number of points in the square should be far larger. But, wait and see. …

Let us take our square to be the unit square, that is, all points on the coordinate plane whose x- and y-coordinates satisfy the inequalities: $0 < x < 1$ and $0 < y < 1$. For the edge, we consider the unit interval, that is all points t such that $0 < t < 1$. From now on we will be a bit lazy, and when we say 'unit square' or the 'unit interval' we will mean the 'set of all points in the unit square' and the 'set of all points in the unit interval'. This laziness will hopefully allow for a more succinct expression, without loss of clarity! Let us denote the unit square as $(0, 1) \times (0, 1)$, and the edge (i.e., the unit interval) as before by $(0, 1)$.

Notice that every point in the square has coordinates (x, y) satisfying the conditions above, so we can represent x and y in terms of their decimal expansions to get $x = 0.x_1x_2x_3\ldots$ and $y = 0.y_1y_2y_3\ldots$. Again assuming that we will consider decimal numbers like $0.5999\ldots$ and $0.6000\ldots$ as being the same, both x and y have unique decimal representations. We then define the function $f\colon (0, 1) \times (0, 1) \to (0, 1)$ by:

$$f(0.x_1x_2x_3\ldots,\ 0.y_1y_2y_3\ldots) = 0.x_1y_1x_2y_2x_3y_3\ldots.$$

You can see that every point in the square has been mapped onto a unique point on the edge, and it is not hard to see that f is an injective function. In other words, starting from the image of a point in the square we can unravel the above process and land back at the original point in the square. Does every point in the unit interval have a partner in the square? No. Consider the following point in the unit interval: $0.909090\ldots$. When we unravel this point using the procedure described above we end up with the point $(0.999\ldots, 0.000\ldots)$. Now we agreed to represent $0.9\ldots$ as 1. So we end up at $(1, 0)$, which is not in the unit square! Similarly the point $0.191919\ldots$ will be mapped onto $(0.1\ldots, 1)$ and the point $0.010101\ldots$ will be mapped onto $(0, 0.1\ldots)$, both of which do not belong to the unit square.

We illustrate this via Figure 2.

While f is injective, it is not a 1-1 correspondence. To prove our result that the unit square and the unit interval have the same cardinality, we resort to the Schroeder-Bernstein Theorem. Since we have an injective function $f\colon (0, 1) \times (0, 1) \to (0, 1)$, we may infer that the cardinality of the unit square is smaller than that of the unit interval. From the fact that we can easily find an injective function from the unit interval to the unit square, the cardinality of the unit interval is smaller than that of the unit square. From Schroeder-Bernstein, we now infer that the unit square and the unit interval have the same cardinality.

Cantor himself was amazed by this result. In fact he spent the years 1871–1874 trying to prove it was false, and finally when he came upon this result, he wrote to his friend, the German mathematician Richard Dedekind, "I see it, but I don't believe it."

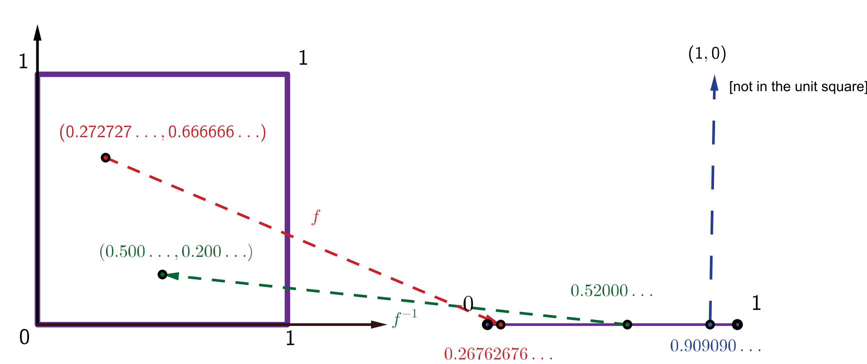

Figure 2

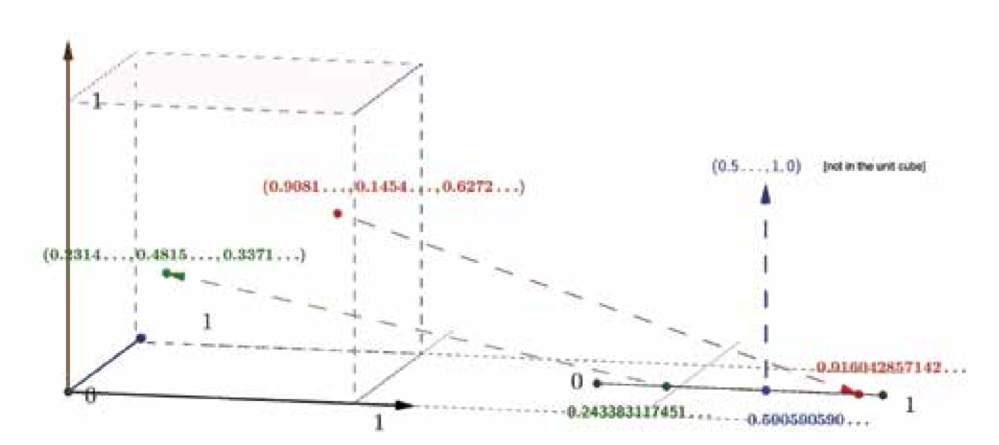

Figure 3

The same technique can be used to show that the cardinality of the unit cube is the same as the cardinality of any one of its edges, or in fact the cardinality of an n-dimensional unit cube is the same as the cardinality of any one of its edges! We illustrate a similar injective function between the 3-dimensional cube and the unit interval in Figure 3.

Cantor once again established, using the Schroeder-Bernstein Theorem, that the cardinality of the unit square is the continuum **c**, as is that of the unit cube. He went on to establish that the cardinality of the points in a 2-dimensional plane, in 3-dimensional space and in n-dimensional space are all the continuum!!

So you can see how hard it is to find a larger infinity. Even going to higher dimensions does not seem to produce a set with greater cardinality. Cantor did a lot of arithmetic with cardinal numbers and showed that if you combine sets by taking unions or cross products, you do not get a new set with larger cardinality. Even if you take the infinite union of infinite sets (remember the infinitely many buses each with infinitely many passengers arriving at the Hilbert hotel), you still do not get a larger infinity! These ideas are rather technical to get into in this article, but one can ask the question: Have we hit the end of the road as far as infinities are concerned? That is, are $\aleph_0$ and **c** the only cardinal numbers that exist? Trust Cantor to prove our intuition wrong yet again. He showed that not only are there more cardinal numbers, there are in fact infinitely many of them!

The hierarchy of infinities

In order to understand the hierarchy of infinities, we need to introduce the idea of a **power set**. Given a set A, the power set of A consists of all the subsets of A. For example if $A = \{1, 2, 3\}$ then

$$P(A) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{A\}\}.$$

Notice that both the empty set ($\{\}$) and the whole set A are elements of $P(A)$. It is a nice exercise to show that if a finite set A has n elements then $P(A)$ has 2^n elements.

It took a Cantor to look at power sets of infinite sets to find larger infinities! In what is now called simply *Cantor's Theorem*, he showed that for any infinite set X, the power set $P(X)$ has a larger cardinality than the set X.

Cantor's theorem

For any infinite set X, $|P(X)| > |X|$.

Proof. $X \subseteq P(X)$, therefore $|X| \leq |P(X)|$. What we now need to show is that it is impossible to have a 1-1 correspondence between X and $P(X)$. As before we will do this by assuming the contrary. Let us begin by assuming that there **is** a 1-1 correspondence between X and $P(X)$. This means that for every element of X we have managed to associate a unique element of $P(X)$, or in other words a unique subset of X, and vice versa.

Let us for the sake of illustration assume that elements of X will be denoted by lowercase letters and elements of $P(X)$ will be denoted by uppercase letters. So:

$$X = \{a, b, c, d, e, f, g, t, \ldots\} \text{ and}$$
$$P(X) = \Big\{X, \{\}, A, B, C, D, E \ldots\Big\}.$$

Suppose (for example) that $C = \{b\}$ and $E = \{a, d, g, t\}$. Let us assume that our 1-1 correspondence associates elements from X to $P(X)$ in the following manner:

$$\begin{array}{ll} a \to X, & b \to \{\}, \\ c \to A, & d \to B, \\ e \to C, & g \to D, \\ t \to E, & \ldots. \end{array}$$

Cantor's genius lay in his ability to realize that there are two kinds of elements in the set X. One kind of element is associated with a set that contains that element itself. For example, a is associated with X, and X contains a. Similarly, t is associated with E, and E contains t. We will call all such elements **Insiders**. The other kind is where the element does not belong to the set with which it is associated. For example, b is associated with the empty set $\{\}$, and clearly $b \notin \{\}$; and e is associated with C, and $e \notin C$. We call all such elements **Outsiders**. Now clearly, every element in X is either an Insider or an Outsider.

Let us collect all the Outsider elements of the set X and call this the *Outsider Subset* of X; we denote it by O. (Notice that O is not empty, because it contains at least β.) Now, if we have a 1-1 correspondence between X and $P(X)$, there must be some element from X associated with O. Let us assume that this element is s. The natural question is, is s an Insider or an Outsider? Suppose that s is an Insider; then by definition, $s \in O$. But O contains all those elements that do not belong to the set with which they are associated, so s cannot belong to O; hence s is an Outsider! Now, suppose that s is an Outsider, then by definition, it should not belong the set with which it is associated, in our case O. But O is the *Outsider Subset* of X and contains all the Outsider elements, so that forces s to be in O, making it (s) an Insider! So you see, we cannot win! If s is an Insider, then it must be an Outsider; and if it is an Outsider, then it must be an Insider!!

This absurd situation arose because we had assumed there was a 1-1 correspondence between X and $P(X)$. It follows that $|P(X)| > |X|$ for any infinite set X. □

I hope the reader appreciates the fact that every time we think we can set up a 1-1 correspondence between a set and its power set, we will produce the *Outsider Subset* and run into the same absurdity we just did.

Cantor not only showed how to produce larger infinities, but also showed how to produce infinitely many infinities! For if we start with X and denote $P(X)$ to be X_1, then by this logic we denote $X_2 = P(X_1)$ and so on. Therefore we then get the following nested sequence of infinite cardinal numbers:

$$|X| < |X_1| < |X_2| < |X_3| \ldots$$

Much more than we bargained for!

The continuum hypothesis

We end this two-part article by explaining where $\aleph_0$ belongs in the hierarchy of infinities and introducing the famous continuum hypothesis.

In this world of infinities, does it even make sense to ask, is there a smallest infinity? Let us start with any infinite set S and take a rather naive approach of creating a subset which is countable. Choose any element s_1 from S. Since S is infinite, $S - \{s_1\}$ is not empty. Choose another element s_2 in this set; clearly, $s_2 \neq s_1$. Again, since S is infinite, $S - \{s_1, s_2\}$ is not empty. Continuing in this vein and choosing an element s_n for each natural number n, we can produce an infinite countable set $T = \{s_1, s_2, s_3, \ldots, s_n, \ldots\}$. Since $T \subseteq S$, we have $|T| \leq |S|$. What our naive approach tells us is that no matter what infinite set S we start with, we have $\aleph_0 \leq |S|$. This tells us that $\aleph_0$ *is the smallest cardinal number*.

Cantor decided to denote the cardinal number just bigger than by $\aleph_0$ by $\aleph_1$, and so on, producing what we would technically call a partially ordered set of cardinal numbers $\{\aleph_0 < \aleph_1 < \aleph_2 \ldots\}$. For Cantor the next obvious question was how to do you get $\aleph_1$ from $\aleph_0$?

What if you look at the power set of the natural numbers. Here is what he found (the modern notation for the cardinality of $|P(\mathbb{N})|$ is $2^{\aleph_0}$):

Another amazing result by Cantor

$$2^{\aleph_0} = \mathbf{c}.$$

That is, there is a 1-1 correspondence between the set of real numbers $\mathbb{R}$ and the set of all possible subsets of $\mathbb{N}$. The proof of this result is beyond the scope of this article, but we urge the interested reader to look up one of the references.

So is $\aleph_1 = \mathbf{c}$? In other words, is $\mathbf{c}$ the next infinity after $\aleph_0$? That, my friend, is the famous continuum hypothesis.

The continuum hypothesis

$$2^{\aleph_0} = \aleph_1.$$

The continuum hypothesis has attracted some of the greatest minds in mathematics. Cantor himself spent the rest of his life trying to prove it. The German mathematician David Hilbert, who was a keen admirer of Cantor (recall the quote from Part I) proposed 23 famous unsolved problems in 1900 during the International Congress of Mathematicians in Sorbonne. These problems have influenced the growth and direction of mathematics to this day. The continuum hypothesis was the first on his list!

The continuum hypothesis has been settled in a strange way, perhaps not to the satisfaction of all. The approach to resolving the continuum hypothesis has been somewhat akin to how the question of whether Euclid's fifth postulate was really needed or not. Remember that in that case, assuming the negation of the postulate led to new non-Euclidean geometries.

In the case of the continuum hypothesis, the famous logician Kurt Gödel showed in 1940 that assuming that the continuum hypothesis is true does not lead to any contradictions assuming the 'standard' axioms of set theory, and Paul Cohen showed in 1963 that assuming that the continuum hypothesis is not true also does not lead to any contradictions. In a sense what Gödel and Cohen showed was that the continuum hypothesis is independent of the standard axioms of set theory.

I hope that you have had a taste of the infinite, and will now be lured to pursue many of its other attributes on your own.

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SHASHIDHAR JAGADEESHAN received his PhD from Syracuse in 1994. He is a teacher of mathematics with a belief that mathematics is a human endeavour; his interest lies in conveying the beauty of mathematics to students and looking for ways of creating environments where children enjoy learning. He may be contacted at jshashidhar@gmail.com.