ClassRoom

A Diagonal Investigation

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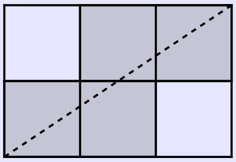
In this issue, we will look at an investigation which at first seems as though it is not going to yield much in terms of patterns. See Figure 1.

Count the squares!

Shown below is a 2×3 rectangle. We see that the diagonal of the rectangle (shown as a dashed line) passes through 4 squares (shown shaded).



Investigate for rectangles of other shapes.

Figure 1

This is a suitable investigation for class 8/9 level and easily accessible for all students. The teacher should provide 1 cm square paper and emphasise the need to use a sharp pencil for this investigation; some children may find it useful to shade the squares through which the

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diagonal passes. A few students will ask if passing through a vertex of a square is considered as 'passing through that square'. The teacher can preempt this by mentioning at the start that the line has to pass through some region of the square (including the boundary) for it to be counted.

With this minimal instruction the teacher should allow the children to proceed with the investigation in whichever way they prefer. Most will begin by drawing different rectangles and counting the number of squares the diagonal is passing through. Some will do it in a random fashion, while others will adopt some kind of a system. Those who are very systematic may start by looking at rectangles that are 1×1, 2×1, 3×1, 4×1, etc., and will quickly make a conjecture that the number of squares that a diagonal passes through is the same as the longer side of a rectangle. *It is important that the students get into the habit of writing down any conjectures or patterns that they observe during the course of an investigation.* Here, as the students look at a different set of rectangles, they will realise that their conjecture is not true; they should also write this down, and if they have any thoughts on how they would now proceed, they should state that as well.

The teacher needs to be in touch with the pulse of the class and must create an ambience free of competition. If required, the teacher could spend some time reiterating that in an investigation, what is important is the thinking process, and the justifications provided for the conjectures, irrespective of whether the conjecture is invalidated with new evidence that may emerge later.

After the students have worked on the investigations for some time, the teacher could conduct a brief discussion on how students could record their results in a precise manner. (See Figure 2.) Some students may suggest that the results could be tabulated, with the width being constant in each table. Others may decide to have one large table with the widths on one side and lengths on the other as shown in Figure 3.

Those who are more adventurous may adopt a symbolic way of presenting the findings, like $s(2,3) = 4$, which states that for a 2×3 rectangle, the number of squares a diagonal passes through is 4. If time permits, one could have a discussion on how length is defined sometimes as the longer side and at other times as the horizontal distance. This is one of the rare instances where mathematicians have not created a convention, a standard way of defining something precisely; and this creates confusion for students and teachers in elementary school. This is probably because it occurs in application-oriented questions, where the everyday usage is also ambiguous.

Pedagogic strategies

This task is designed to give students a feel of how mathematicians 'work'. It develops the skills of documentation, communication, reasoning and conjecture.

Figure 2

$L \backslash W$	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2			4		6		8	
3				6	7		9	10
4					8		10	
5						10	11	12
6							12	
7								14
8								

Figure 3

As the investigation proceeds, some students are able to see that $s(W,L) = W + L - 1$ in some instances. A teacher can nudge them towards sorting rectangles that follow the above rule and those that do not. Those who are good at spotting visual patterns will see that in those rectangles that do not fit the above rule, the diagonal sometimes passes through the vertices of some squares, whereas in the rectangles that follow the rule, the diagonal never passes through the vertices, except at the start and the end. Most children will also come across the more specific case of $W = L$, i.e., the square, and find that in this case the number of squares a diagonal passes through is the same as a side of a square; also, the diagonal passes through the vertices of all the squares that it touches. One may also find students following a more analytical approach, for example starting with a rectangle and seeing what happens when the rectangle is enlarged by different integral scale factors e.g. $s(2,3)$, $s(4,6)$, $s(6,9)$. The recording sheet could also have space for them to document their approach; this will help students to formalise their thinking.

This is an investigation that can be started in class, and then left for students to continue at home. The teacher can spend a few minutes towards the end of the class to discuss some of the approaches and conjectures that students have come up with, to enable everyone to proceed with some degree of confidence. If there are students who have seen the general case or are close to it, the teacher could ask them to justify their answer and if they are keen, to extend the investigation to three dimensions by looking at cuboids. The teacher should, however, not reveal the general case to students who are still grappling with the original problem, and should also advise those who have come upon the answer to not give it to the others.

After a week, the teacher could follow up on the investigation by giving students an opportunity to share their findings with the rest of the class. The teacher needs to allow this to happen in a calibrated way, so that a range of different approaches and patterns are revealed before coming to the general case. There will be students who will have explored rectangles with dimensions that are only prime numbers and would state that in those cases the $W + L - 1$ formula works. The teacher can encourage this to be stated in symbolic form, e.g., $s(W, L) = W + L - 1$ where both W, L are prime numbers; this could also be extended for W, L being coprime. Some students would have seen that if the rectangle is of the kind where one side is a multiple of the other, i.e., $L = kW$ where k is some positive integer, then $s(W, L) = L$ (this also works for the case when $k = 1$, a square). These formulas then lead to the general case $s(W, L) = W + L - \textsc{gcd}(W, L)$. (Here, GCD means *greatest common divisor* which is the same as *highest common factor*.)

The first step in seeing why this relation holds lies in spotting the relation between $s(W, L)$ and $s(kW, kL)$ where k is any positive integer. For example, consider the case when $(W, L) = (2, 3)$ and $k = 2$. Examining a 4×6 rectangle, we see that its diagonal passes through two 2×3 rectangles arranged corner-to-corner as shown in Figure 4. We infer from this that $s(4, 6) = 2 \times s(2, 3)$.

In much the same way we see that for any positive integer k,

$$s(kW, kL) = k \times s(W, L).$$

The second step is to understand how to deduce the general formula from the conjectured formula for the special case when W, L are coprime, i.e., $s(W, L) = W + L - 1$. Suppose that W and L are not coprime; say $\textsc{gcd}(W, L) = k$ (where $k > 1$). Then W/k and L/k are coprime positive integers, hence:

$$s\left(\frac{W}{k}, \frac{L}{k}\right) = \frac{W}{k} + \frac{L}{k} - 1.$$

Multiplying through by k we get:

$$k \times s\left(\frac{W}{k}, \frac{L}{k}\right) = W + L - k.$$

But we know that

$$k \times s\left(\frac{W}{k}, \frac{L}{k}\right) = s(W, L).$$

Hence $s(W, L) = W + L - k$, i.e.,

$$s(W, L) = W + L - \textsc{gcd}(W, L).$$

So the proof entirely hinges on showing that if W, L are coprime, then

$$s(W, L) = W + L - 1.$$

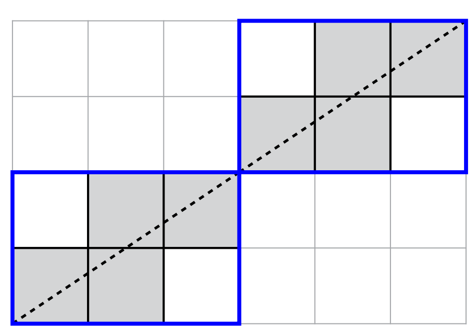

Figure 4

Devising the proof that this is so is a lovely exercise. With careful prompting, the teacher should be able to coax the proof from the class.

Remarks.

(1) For those pursuing the investigation for cuboids, straws and connectors could be provided and also multi-link cubes to help in visualising the problem. The 3D problem has a lot in common with the 2D version.

(2) Students study the notion of highest common factor (GCD) in school, but rarely come across it elsewhere; the unexpected appearance of it in this context should create an *AHA!* moment for the students.

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