The Digital Root

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Introduction

The *digital root* of a natural number n is obtained by computing the sum of its digits, then computing the sum of the digits of the resulting number, and so on, till a single digit number is obtained. It is denoted by $B(n)$. In Vedic mathematics, the digital root is known as *Beejank*; hence our choice of notation, $B(n)$. Note that the digital root of n is itself a natural number. For example:

- $B(79) = B(7+9) = B(16) = 1+6 = 7$; that is, $B(79) = 7$.
- $B(4359) = B(4+3+5+9) = B(21) = 3$.

The concept of digital root of a natural number has been known for some time. Before the development of computer devices, the idea was used by accountants to check their results. We will examine the basis for this procedure presently.

Ten arithmetical properties of the digital root

The following arithmetic properties can be easily verified. Here $n, m, k, p, \ldots$ denote positive integers (unless otherwise specified).

Property 1. *If* $1 \le n \le 9$, *then* $B(n) = n$.

Property 2. *The difference between* n *and* $B(n)$ *is a multiple of* 9; *i.e.,* $n - B(n) = 9k$ *for some non-negative integer* k.

To prove this, it suffices to show that for any positive integer n, the difference between n and the sum $s(n)$ of the digits of n is a multiple of 9. This step, carried forward recursively, will prove the claim. But the claim is clearly true, for if

$$n = a_0 + 10a_1 + 10^2a_2 + 10^3a_3 + \cdots,$$

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then $s(n) = a_0 + a_1 + a_2 + a_3 + \cdots$, and therefore:

$$n - s(n) = (10-1)a_1 + \left(10^2 - 1\right) a_2 + \left(10^3 - 1\right) a_3 + \cdots.$$

Since the coefficients $10-1,\ 10^2-1,\ 10^3-1, \ldots$ are all multiples of 9, it follows that $n - s(n)$ is a multiple of 9.

Corollary: *The difference between n and $B(n)$ is a multiple of 3.*

Property 3. *If n is a multiple of 9, then $B(n) = 9$. If n is not a multiple of 9, then $B(n)$ is equal to the remainder in the division $n \div 9$.*

This follows from Property 1 and Property 2.

Property 4. *For all pairs m, n of integers, the following relations are true:*

$$B(m+n) = B\left(B(m) + B(n)\right),$$
$$B(m\,n) = B\left(B(m)\,B(n)\right).$$

For example, let $m = 12$, $n = 17$. Then $B(m) = B(12) = 3$ and $B(n) = B(17) = 8$. Also, $m + n = 29$, $mn = 204$, so $B(m+n) = 2$, $B(mn) = 6$. Now observe that:

$$B\left(B(m) + B(n)\right) = B(3+8) = B(11) = 2 = B(m+n),$$

and

$$B\left(B(m)\,B(n)\right) = B(3 \times 8) = B(24) = 6 = B(mn).$$

The two relations may be proved in general using Property 3. They will follow from the following more general assertion: If integers m' and n' are such that m, m' leave the same remainder under division by 9, and n, n' likewise leave the same remainder under division by 9, then

$$B(m+n) = B\left(m' + n'\right),$$
$$B(m\,n) = B\left(m'\,n'\right).$$

We leave the proof of this to you; it follows once again from Property 3.

Comment. It is Property 3 which forms the basis of the traditional method of "checking a calculation by casting out nines." Thus, we can quickly show that the computation

$$34567 \times 23456 = 810802552$$

must be incorrect, because the digital root of the expression on the left side is $B(7 \times 2) = 5$, while the digital root of the expression on the right side is 4.

Property 5. *A prime number exceeding 3 cannot have a digital root equal to 3, 6 or 9.*

For, since $n - B(n)$ is a multiple of 3, if $B(n)$ is a multiple of 3, then so must be n; and the only prime number which is a multiple of 3 is 3 itself.

Remark: The following statement looks plausible but is **not** true: "The greatest common divisor of n and $B(n)$ is equal to the greatest common divisor of $B(n)$ and 9." We invite you to find a counterexample for this claim.

Property 6. *The digital root of a triangular number is one of the following numbers:* $1, 3, 6, 9$.

Recall that a **triangular number** has the form $m(m+1)/2$ where m is a positive integer. To prove the above claim, we consider the different forms that a triangular number $n = m(m+1)/2$ can have, depending on the remainder that m leaves under division by 6.

The different forms for m are: $6k$, $6k+1$, $6k+2$, $6k+3$, $6k+4$, $6k+5$. If $m = 6k$ or $6k+2$ or $6k+3$ or $6k+5$, then the product $m(m+1)/2$ is a multiple of 3; hence the digital root of $m(m+1)/2$ will be one of the numbers $3, 6, 9$. If $m = 6k+1$ or $6k+4$, then $m(m+1)/2$ is of the form $9m+1$; hence its digital root is 1.

Expressed negatively, the above result yields a useful corollary.

Corollary. A triangular number which is not a multiple of 3 has digital root equal to 1.

Property 7. *If n is a perfect square, then* $B(n) \in \{1, 4, 7, 9\}$.

We consider the different forms that a perfect square $n = m^2$ can have, depending on the remainder that m leaves under division by 9. The different forms for m are: $9k$, $9k \pm 1$, $9k \pm 2$, $9k \pm 3$, $9k \pm 4$. By squaring the expressions and discarding multiples of 9, we find that $B(n) \in \{1, 4, 7, 9\}$ in each case.

Property 8. *If n is a perfect cube, then* $B(n) \in \{1, 8, 9\}$.

The same method may be used as in the case of the squares.

Property 9. *If n is a perfect sixth power, then* $B(n) \in \{1, 9\}$.

If n is a perfect sixth power, then it is a perfect square as well as a perfect cube; hence $B(n) \in \{1, 4, 7, 9\}$ as well as $B(n) \in \{1, 8, 9\}$. This yields: $B(n) \in \{1, 9\}$. (A nice application of set intersection!)

Corollary. A perfect sixth power which is not a multiple of 3 has digital root equal to 1.

Property 10. *If n is an even perfect number other than* 6, *then* $B(n) = 1$. Recall that a **perfect number** is one for which the sum of its proper divisors equals itself. The first few perfect numbers are: 6; 28; 496; 8128; 33550336. The digital roots of these numbers are:

$$B(28) = B(10) = 1,$$

$$B(496) = B(19) = 1,$$

$$B(8128) = B(19) = 1,$$

$$B(33550336) = B(28) = 1,$$

and so on.

This result is far from obvious and will need to be proved in stages. The full proof is given in the addendum at the end of this article.

Concluding remark. The notion of digital root has been known for many centuries. As described in this article, there is a simple number theoretic basis for the notion. The simplicity of the topic makes it an attractive one for closer study by students in middle school and high school. It certainly needs to be better known than it is at present.

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