ClassRoom

The Difference-of-Two-Squares Formula: A New Look

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The difference-of-two-squares formula $a^2 - b^2 = (a-b)(a+b)$ is so basic that it would seem a difficult task to say anything new about it! But **Agnipratim Nag** of Frank Anthony Public School, Bangalore (Class 8) has done just this. In this short note we describe his interesting and innovative approach to prove the identity. It has particular relevance for those who teach at the middle school level.

The proof assumes throughout that a and b are integers. Pedagogically, this is very appropriate, as the difference-of-two-squares identity should be introduced to students as a relationship between integers. Interesting explorations involving integers can be designed which will lead to students to the identity. The extension to arbitrary numbers can happen later. In the account below, a is assumed to be the larger number.

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The case when a and b are consecutive integers, i.e., a $=$ b $+ 1$. We have:

$$\begin{aligned} ab &= ba, \\ \therefore\ a(a-1) &= b(b+1), \\ \therefore\ a^2 - a &= b^2 + b, \\ \therefore\ a^2 - b^2 &= a + b. \end{aligned}$$

This relation may be expressed in words as follows:

The difference between the squares of two consecutive integers is equal to the sum of the two integers.

The relation $a^2 - a = b^2 + b$ can be depicted in an attractive way (Figure 1) when we recall that a^2 and b^2 are a pair of consecutive squares.

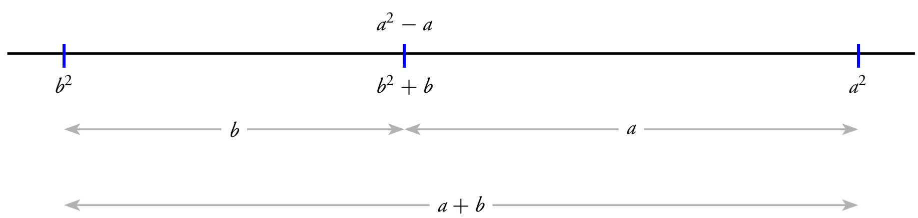

Figure 1

The case when a and b are not consecutive integers. There are two ways of proceeding. The more symbolic approach is this. Let $a - b = k$; then $a = b + k$, $b = a - k$, and:

$$\begin{aligned} ab &= ba, \\ \therefore\ a(a-k) &= b(b+k), \\ \therefore\ a^2 - ak &= b^2 + bk, \\ \therefore\ a^2 - b^2 &= ak + bk = k(a+b), \\ \therefore\ a^2 - b^2 &= (a-b)(a+b). \end{aligned}$$

A more interesting and also more colourful approach (as described by Agnipratim) is the following. The integers strictly between b and a are $b+1, b+2, \ldots, a-2, a-1$. Let these integers be denoted by $p_1, p_2, \ldots, p_{k-2}, p_{k-1}$ (if $a - b = k$, then there are $k - 1$ integers strictly between b and a); so we have $p_1 = b + 1, p_2 = b + 2, \ldots, p_{k-1} = a - 1$. The configuration may now be depicted as shown in Figure 2.

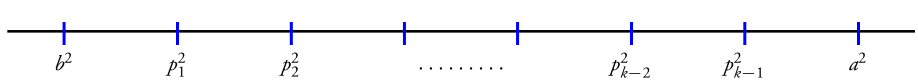

Figure 2

(Note: The figure is meant to be symbolic. It may give the impression that the gaps between $a^2, p_1^2, p_2^2, \ldots$ are all the same, but this is not actually the case. In fact, the gaps steadily get larger as we move from left to right.)

The total gap between a^2 and b^2 is clearly equal to the sum of the gaps between a^2 and p_{k-1}^2, between p_{k-1}^2 and p_{k-2}^2, …, between p_2^2 and p_1^2, and between p_1^2 and b^2:

$$a^2 - b^2 = \left(a^2 - p_{k-1}^2\right) + \left(p_{k-1}^2 - p_{k-2}^2\right) + \cdots + \left(p_2^2 - p_1^2\right) + \left(p_1^2 - b^2\right).$$

Since $\{a, p_{k-1}\}$, $\{p_{k-1}, p_{k-2}\}$, …, $\{p_2, p_1\}$, $\{p_1, a\}$ are pairs of consecutive integers, the result established earlier applies ("The difference between the squares of two consecutive integers is equal to the sum of the two integers"). Hence

$$\begin{aligned} a^2 - p_{k-1}^2 &= a + p_{k-1}, \\ p_{k-1}^2 - p_{k-2}^2 &= p_{k-1} + p_{k-2}, \\ p_2^2 - p_1^2 &= p_2 + p_1, \\ p_1^2 - b^2 &= p_1 + b, \end{aligned}$$

which means that

$$a^2 - b^2 = (a + p_{k-1}) + (p_{k-1} + p_{k-2}) + \cdots + (p_2 + p_1) + (p_1 + b)\,.$$

The expression on the right may be rearranged to give:

$$a^2 - b^2 = (a + b) + 2\,(p_1 + p_2 + \cdots + p_{k-2} + p_{k-1})\,.$$

A further rearrangement is possible, but the behaviour is slightly different depending upon whether k is odd or even. We illustrate the behaviour using two specific values, $k = 5$ and $k = 6$. Consider first $k = 5$; we have:

$$\begin{aligned} a^2 - b^2 &= (a + b) + 2\,(p_1 + p_2 + p_3 + p_4) \\ &= (a + b) + 2\,(p_1 + p_4) + 2\,(p_2 + p_3)\,. \end{aligned}$$

Now we have, clearly:

$$p_1 + p_4 = a + b, \quad p_2 + p_3 = a + b.$$

Hence:

$$a^2 - b^2 = (a + b) + 2(a + b) + 2(a + b) = 5(a + b),$$

and since $a - b = 5$ in this instance,

$$a^2 - b^2 = (a - b)(a + b).$$

Next, consider $k = 6$; we have:

$$\begin{aligned} a^2 - b^2 &= (a + b) + 2\,(p_1 + p_2 + p_3 + p_4 + p_5) \\ &= (a + b) + 2\,(p_1 + p_5) + 2\,(p_2 + p_4) + 2p_3. \end{aligned}$$

As earlier, we have:

$$p_1 + p_5 = a + b, \quad p_2 + p_4 = a + b, \quad 2p_3 = a + b.$$

Hence:

$$a^2 - b^2 = (a + b) + 2(a + b) + 2(a + b) + (a + b) = 6(a + b),$$

and since $a - b = 6$ in this instance,

$$a^2 - b^2 = (a - b)(a + b).$$

This reasoning clearly holds for all values of k. We have not bothered to write out the argument for the general case in a formal manner, but the way the terms can be rearranged to add up to $a + b$ should be clear. We deduce that the relation

$$a^2 - b^2 = (a - b)(a + b)$$

is always true, i.e., it is an identity. □

Figure 3 shows a photograph of a page from Agnipratim's notebook.

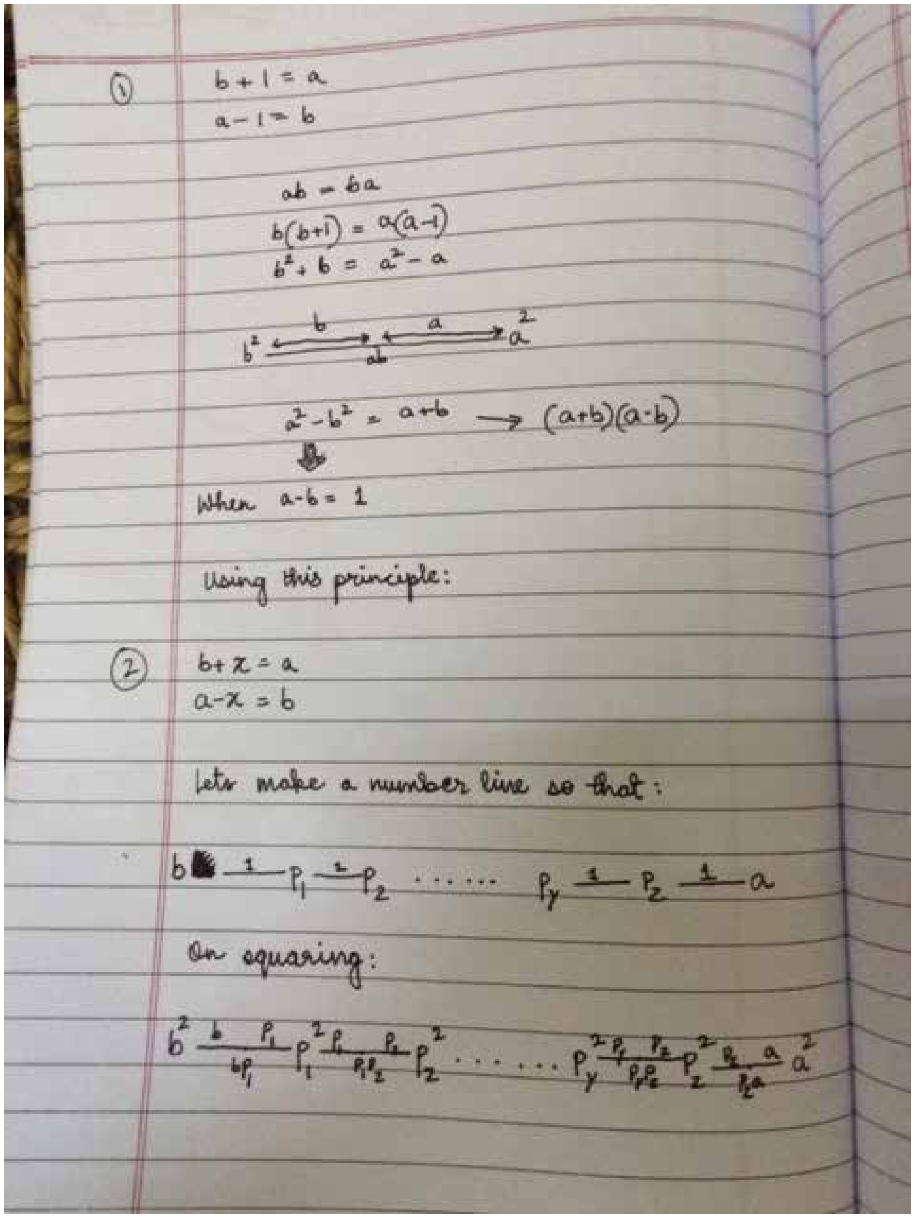

Figure 3. Image sent to us by Agnipratim Nag

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